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Abstract

Large institutions like banks, hedge funds or insurance companies are showing so much interest in the cryptocurrency market. This emerging market showed phenomenal returns during the last ten years. However, it comes with a price: volatility and unpredictability of the price which can make those institutions insolvent in a blink of an eye if they cannot manage the risks.

In the scope of this project, we try to understand this relatively new-born market to hedge its multiple risks. We have studied the properties of the emerging Bitcoin derivatives market, mainly the different actors, the volume, the bid-ask spread ... to assess information that can be useful during the modelling process. For the first part we used several standard stochastic models, and showed why one should add complexity to the model varying from stochastic volatility to jumps. The last model implemented was the stochastic volatility with correlated jumps model that groups all characteristics of the previous ones. This model showed relatively good results. In the calibration process, various techniques were used varying from solving optimization problems to MCMC methods. We have later on shown that both the price and volatility processes of Bitcoin exhibit multi-fractality using the multi-fractal detrended fluctuation analysis and wavelets analysis. Those two methods revealed that the bitcoin market follows the fractal market hypothesis theory and that the process presents persistence or roughness from time to time. Thus, we considered a rough volatility model and tried to calibrate it using neural networks. Finally, we reviewed the regulations proposed by the Basel committee.

Résumé

Les grandes institutions comme les banques, les hedge funds ou les compagnies d'assurance montrent un grand intérêt pour le marché des crypto-monnaies. Ce marché émergent a montré des rendements phénoménaux au cours des dix dernières années. Cependant, ceci a un prix : la volatilité et l'incertitude du prix qui peuvent rendre ces institutions insolvables en un clin d'œil si elles ne peuvent pas gérer les risques.
Dans le cadre de ce projet, nous essayons de comprendre le maché du Bitcoinn pour couvrir ses multiples risques. Nous avons étudié les propriétés du marché des dérivés de Bitcoin, principalement les différents acteurs, le volume, le spread bid-ask... afin d'évaluer les informations qui peuvent être utiles lors du processus de modélisation. Pour la première partie, nous avons utilisé plusieurs modèles stochastiques standards, et avons montré pourquoi il fallait ajouter de la complexité à chaque modèle. Le dernier modèle implémenté était le modèle de volatilité stochastique avec sauts corrélés qui regroupe toutes les caractéristiques des modèles précédents. Ce modèle a montré des résultats relativement bons. Dans le processus de calibration, diverses techniques ont été utilisées, allant de la résolution de problèmes d'optimisation aux méthodes MCMC. Nous avons ensuite montré que les processus de prix et de volatilité du bitcoin présentent une multi-fractalité en utilisant l'analyse des fluctuations tendues multi-fractales et l'analyse des ondelettes. Ces deux méthodes ont révélé que le marché du bitcoin suit la théorie de l'hypothèse de marché fractale et que le processus présente une persistance ou une rugosité de temps en temps. Ainsi, nous avons considéré un modèle de volatilité rugueuse et avons essayé de le calibrer en utilisant des réseaux de neuronaux. Enfin, nous avons examiné les règlements proposés par le comité de Bâle.

Contents

Chapter 1

Introduction

We can trace back crypto-assets to the creation of Bitcoin by a mysterious researcher surnamed "Satoshi Nakamoto" in the year 2008. Since then, the blockchain technology and cryptocurrencies have gained much interest for many investors. The emergence of Bitcoin options and futures in cryptocurrency derivatives exchanges have announced a new era in dealing with Bitcoin and other currencies risk hedging problems. Since the market crash of 1987, these tools gained a real interest as investors were in the need of better strategies to protect their portfolios. Options and futures give a certain freedom to trade and hedge volatile fluctuations in the asset price effectively. As opposed to other financial markets, a little research was done on this growing market Which presents a major hurdle for institutions that want to invest or already investing in Bitcoin like Grayscale, Ark Invest, Tesla ...

In the scope of this report, we investigate Bitcoin dynamics to better hedge the risk of investing in this new-born market. Our approach consists of implementing different stochastic models depending on the complexity of the market to essentially better understand Bitcoins dynamics and its derivative market and fit its volatility surface. The studies done will help us derive some risk management strategies and see if the regulations that are put on place are sufficient for cryptocurrency products.

In the forthcoming chapter, we describe the cryptocurrency derivatives market. We compare its characteristics with more usual financial markets like the Equity, Fx or Commodities markets. And, therefore, try to derive some conclusions that might help us in the modelling process.

In the third chapter, we define some stochastic processes that we would use in the modelling process and some risk management measures that will come in handy.

In the scope of the fourth and the fifth chapter we start by implementing some classical models starting from the most basic ones and trying to add some complexity to the model used each time to better fit the volatility surface. We try also to compare the results we find to other financial markets. In the second part, we prove the existence of a fractal behaviour in the Bitcoin market and try to come up with a rough model that fits the crypto-assets changing dynamics, calibrate it and discuss the results.

In the last chapter, using the conclusion elaborated in the previous chapters, we review the regulations put on place at the moment of the redaction of this thesis and present the different vues of the Bale committee and the financial associations.

Chapter 2

Cryptocurrencies Market

"God understands more about the financial markets than many who write about them."

Jean-Claude Juncker

Introduction

Bitcoin was known to a tiny group of early adopters back in 2008. The most convenient method to own this currency was to mine[1]it on computer. As Bitcoin grew, exchanges were created to ease its transfer. Over time, these exchanges were joined by increasingly efficient financial institutions that offered improved functionality to their clients. Currently, the Bitcoin market has sufficient liquidity for almost all traders, and new products are constantly being added to product offerings. Over time, the exchanges were backed by more and more efficient financial institutions. Right now, the Bitcoin market is liquid enough for all investors and traders. Thus more investors are being interested in this market and new financial products are being created overtime.
A growing market for sure but most importantly a new one to explore and to investigate, as with looking into how it works, we can see how agents act to market changes. Bitcoin being a decentralized currency that is priced by supply and demand makes information available. In a matter of fact, being related to blockchain, one can have a lot of information about this currency. For instance, one can have insight on its supply looking into the Hash rate[2] information .Therefore, several agents assume its market to be efficient[3] thus quickly reacting to public information. Yet seeing that bubbles and crashes can exist and that some

[1]Mining is the process of validating a transaction, such as in bitcoins, by encrypting the data and storing it in a blockchain.

[2]

[3]EFH states that investors in the market act rationally and that the stock price indicates all the relevant information.

investors were able to consistently beat the market, we might doubt this strong hypothesis. This might also be justified by the fact that much of the investors are new to trading and therefore are not acting rationally yet based on mass psychology driven by fear and greed.

2.1 Bitcoin Markets' Actors

Bitcoin miners are crucial participants. They verify the legitimacy of Bitcoin transactions and revise the public ledger[4] using the Proof-of-Work[5] system. Bitcoin is "mined" by powerful computers that ensure not only security but also transaction transparency. Miners are rewarded in bitcoin according to the halving principle[6] . Bitcoin mining is computationally intensive. Therefore, it requires an important financial investment. We are witnessing recently larger mining operations taking over the mining sphere and individual miners colliding to form mining pools. See fig 2.1.

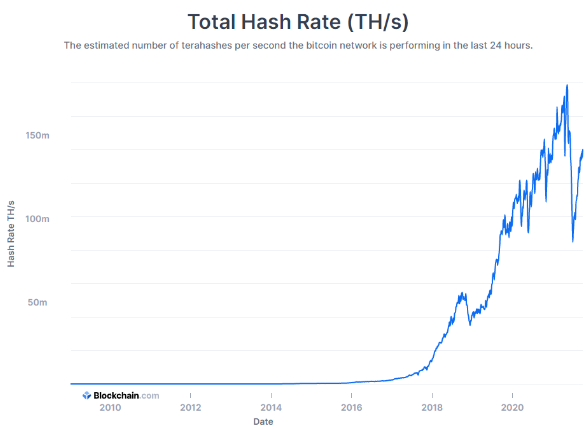


Figure 2.1: Total bitcoin's hash rate

Retail traders and investors Retail traders and investors are increasingly interested in the Bitcoin market. Particularly because they see in Bitcoin an interesting hedge against global economic uncertainty. Bitcoin retail investors buy cryptocurrency hoping to make considerable long-term profit based on this investment long-term growth potential. They generally do not actively trade bitcoin. But rather retain it as its value grows. Nevertheless, they attempt to realize short-term benefits through daily, weekly and monthly trades.
Institutional Investors and Traders Institutional traders and investors are however quite relucant to invest in Bitcoin. Mainly due to its volatility and relatively low market

[4]The public ledger traces participants' identities and cryptocurrency balances in a (pseudo-)anonymous form. It transcripts all the network transactions.

[5]Proof of work (PoW) is a form of cryptographic process where the miner (prover) creates money by solving a complex mathematical calculation

[6]Bitcoin halving refers to cutting in half the reward obtained after successfully mining a block. It also halves Bitcoin's inflation rate as well as the rate at which new bitcoins enter circulation

capitalization. Nevertheless, they significantly increased their adoption in recent years, with hedge funds and quant traders from major investment banks (JP Morgan, Goldman Sachs,etc..) leading the way. Several asset managers, and institutional investors, including banks are now recommending their clients to invest 5% of their wealth in Bitcoin.
Corporations Corporations response to Bitcoin is highly contrasted. Several corporation are considering Bitcoin as an investment opportunity in employees' retirement plans, while others are implementing Bitcoin-compatible payment systems.
Some corporations and large investors are holding massive amount to the point of being called whales. This referring to their capacity to move the Bitcoin prices when they trade large amounts.
Exchange An exchange acts as a market intermediate between two parties. Those who want to buy and those who want to sell. By displaying the bids and asks of both parties, it facilitates trades. When a buyer and a seller name the same bid and ask respectively, the trade is executed and the exchange collect a fee. Currently, most exchanges operate as custodians. In fact, users need to deposit bitcoin at the exchange and withdraw it only once they finish trading Meanwhile, the exchange has full control over the cryptocurrency which can lead to high security risks.

Different exchange markets

- **Chicago CME** The CME was among the first to lauch Bitcoin derivatives it opened its bitcoin futures platform on the 18^{th} if December 2017. In addition to bitcoin contracts, the exchange offers micro Bitcoin futures that represent one of tenth the size of a standard contract.

- **Coinbase** is a crypto company based in San Francisco, and backed by trusted investors. It is considered the largest exchange and broker.

- **Deribit** The platform is one of the most famous exchange platforms, hosting futures and options markets on both Bitcoin and Ethereum. Yet, it is not regulated.
  All hosted options are European-style. The minimum order size is respectively 0.1 contract and 1 contract on Bitcoin and Ethereum. Markets operate 24/7. There are some fees to the deribit users and to the market makers that lay within *4.5 bp* and *5.5 bp* These fees can be neglected later on during the quantitative analysis.

- **Binance** fee's system is progressive meaning the commission is inversly proportionate to one's trading volume. As for regulations, Binance as a group is not overseen by a specific licensing body yet it is backed by wings that are regulated in almost each country. Binance's trading volumes are huge yet the derivatives' market is still developing.

2.2 Bitcoin's price

To understand Bitcoin's price let's try to answer simpler questions first: Why does Bitcoin have value ?
Bitcoin is actually scarce meaning that there will never be more than 21 million bitcoin. Unlike fiat currencies, it is immune to quantitative easing and other inflationary measures. It has no leaders or voting. This protects it from malicious changes and political capture. Finally, as a digital currency, Bitcoin can be easier to transport, store, and divide, solving some of gold's weaknesses. The Bitcoin price, like any other commodity and because of it's decentralised nature, is determined through supply and demand. Governments cannot control it, yet one must say that they can influence their citizens' decisions about the currency and thus influencing demand. Bitcoin mining is though altered by changes in energy price and the accessibility and price of computers and mining tools and thus influencing supply.
This brings up a second question which is how the price of bitcoin is determined and why does it differ from an exchange to another?
Bitcoin is traded 24 hours a day 7 days a week. Its heart never ceases. Every minute bitcoin's heart beats again and therefore there is no closing price. Its value is determined upon a rolling average. That means that given that there is no global standard for the price of bitcoin, investors cannot be sure that Google, a digital currency exchange, or another price tracker is accurate since most price trackers calculate an average estimate on the price of bitcoin, traded recently, based on the trading book of a trusted bitcoin exchange. For instance, Google bases its figures off of the Coinbase API, that explains the fact that it directly converts the value of bitcoin to the U.S. dollar.

Finally, the level of supply and demand may vary across different exchanges. Hence, the difference observed in Bitcoin price. It's true that if the price on one exchange is significantly lower than on another, might shift the supply and demand levels and thus harmonizing the price on different markets. However the problem is that moving money across exchanges can be inefficient as the cross fees can be exorbitant, especially for small transactions. Which means, it's hard for traders to arbitrage differences across exchanges, that allows the persistence of those price differences longer than that would in a more efficient market.

2.3 The derivatives market

The derivatives market on cryptocurrencies is exponentially growing as we see a rising open interest for those contracts[7]. See fig 2.2.
Crypto options can be traded on the Chicago Mercantile Exchange yet most of the trades are taken place on unregulated exchanges. The most known Deribit and Binance.

[7]Open interest is the total number of outstanding contracts that are held by market participants at the end of each day. Open interest measures the total level of activity into the futures market.

Deribit options are European with either BTC or ETH as an underlying. The price in USD is determined by using the latest futures prices. As opposed to standard financial market where we can find exotic and synthetic derivatives, Crypto derivatives market contains [8]

- Regular futures: Contracts consisting of delivering an underlying at an expiry date.
- Non-deliverable forwards: Futures with no physical settlement, settled by the gap between the spot and forward strike price.
- Perpetual futures (perpetual swaps): are futures that have no expiry date. Traders can keep an position open without rolling one contract into another. These contracts are kept in line with the spot rate with a funding-rate mechanism: At any given interval, a difference is paid or received depending on whether the instrument is trading above the spot or below.
- Leveraged tokens: Assets that offer a certain exposure to cryptocurrencies. The leverage can be either fixed or floating (the case of Binance). Similar to leveraged ETFs[9], this type of contracts allows to make big bets without collateral or margin requirements.
- Options: A retail product that can offer a bullish or bearish exposure by paying the positive or negative difference between a the underlying and a strike at an expiry date.

2.3.1 Crypto futures

Selling or buying a futures contract is actually giving a bearish or bullish prediction about the future price of an asset. Taking a position in a future is the speculation on whether one thinks the contract will expire above or below the already fixed strike. Bitcoin futures come with various benefits.

The major interest in futures comes from miners for a hedging purpose. Just like standard farmers, miners invest an important capital in their production process and have long term commitments to the blocks mining on the blockchain, in return reaping rewards.

However, the fluctuating price of bitcoin poses a significant problem. the returns of the mining can be very unpredictable. Therefore, miners have to use futures to protect themselves as they have fixed running costs and having costs exceeding returns for a long period of time can lead to their insolvency.

Another interest is that futures offer a low cost of short selling[10]. A really low settlement fee is just paid in advance. For example, Kraken charges 0.05% for fixed maturity contracts. Expiry of futures is usually 3 months but can go to 6 months or 1 Week.

[8](*How Derivatives Amp Up Already Heady Crypto Markets*, n.d.)

[9]exchange-traded funds

[10]They are better than CFDs (contract for differences) because there is no need to pay overnight financing cost

The strike price is the price of the futures contract set by the markets. an investor, thus, have to determine whether the price of Bitcoin will finish above or below the strike. The strike itself is actually determined on a demand and supply basis and it is the price that markets believe Bitcoin will be worth at the expiry.

Finally, the major difference between options and futures is that options allow for market speculation while futures are predominantly used for risk management. Options also carry low risk compared to futures, since the maximum risk for an option is the premium paid, whereas the risk in a futures contract is limitless.

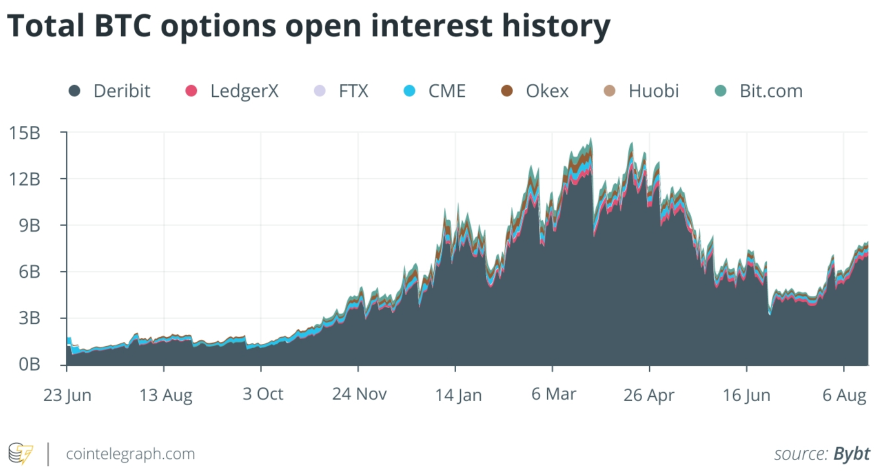


Figure 2.2: Total bitcoin options open interest

2.4 Market schedule

A major difference between cryptocurrency and other financial markets is the 24-hour market structure. Investors, therefore, have to think about the daily price change in their positions through different lenses then the ones they are used to. For instance, taking the case of the New York Stock Exchange, Market trade during regular market hours[11]. A consequence of this is the existence of opening and closing prices. To compute the price change in a given stock, one subtracts the current price from closing price of the previous day and then divide the spread by the latter one. Since no such closing time exists for cryptocurrency markets, the aforementioned method can't be used. The workaround in place is to compute the 24-hour price change by comparing the current market price to the one that was trading the day before at exactly the same time. The problem is that the denominator, which represents yesterday's closing price for usual markets and stays the same all day, keeps changing throughout the day for cryptocurrency.means that if you're

[11]After-market hours do exist for these markets but generally the liquidity is thinner and prices executions are less favorable.

just looking at the percentage change over the last 24 hours, you can't tell whether you're seeing real-time price movement in the cryptocurrency or just residual price volatility from the day before.

2.5 Liquidity: Spread and volume

In the fast-paced cryptocurrency markets, every quantitative analyst should grasp the concept of liquidity while studying the market. Liquidity explains how easy a particular asset can be exchanged without altering the stability of its price. A high liquidity is and indication of a vibrant and stable market. As a matter of fact, in a sufficiently liquid market, trades can be executed easily and most importantly at fair prices. Due to the infancy of cryptocurrencies, the market is considered not liquid. It cannot absorb large orders without changing the value of its assets. Illiquidity is actually tightly related to volatility since anyone with a large order can easily disrupt or manipulate cryptocurrency price.
But how can we measure bitcoins derivatives market liquidity ? Well to answer that one must look into two indicators the bid-ask spread and the volume of transactions.

2.5.1 The volume

We can see in fig 2.3 that 50-days average trading volume for bitcoin options is 639 Mil dollars which represents nothing compared to the S&P500 where the 50-days options average trading volume exceeds 60 Billion dollars.
Greater trading volume means greater trading activity and is therefore a strong indicator of a liquid market. Plus a higher volume usually can back up the fact that a price movement is actually a trend and not a noise. Finally an important trading volume makes price distortion very hard if not impossible and thus the traded prices are the markets' fair prices. In fact, when sudden spikes are observed in the price, the asset price might be manipulated by whales due to the limited volume.

2.5.2 The bid-ask spread

In this part we investigate the derivatives market at its microscopic level. The bid-ask spread is the difference between the price in the order book at which market agents are willing to sell or buy a derivative. When studying derivatives' markets, the spread can be used to identify liquid contracts. Derivatives markets tend to be more fragmented, with contracts of many expiration dates and strike prices, resulting in a significant number of very low liquid contracts with wide spreads. As we can see in fig 2.5, the spread for a call option almost at the money is almost 1300 *bp* which is considered to be enormous compared to the For-ex market where the spread of EUR/USD doesn't exceed 5 *bp*. Often quotes in the order book come from market makers. To freshen up one's memory, market makers are companies or desks within companies that quote buy and sell prices of financial

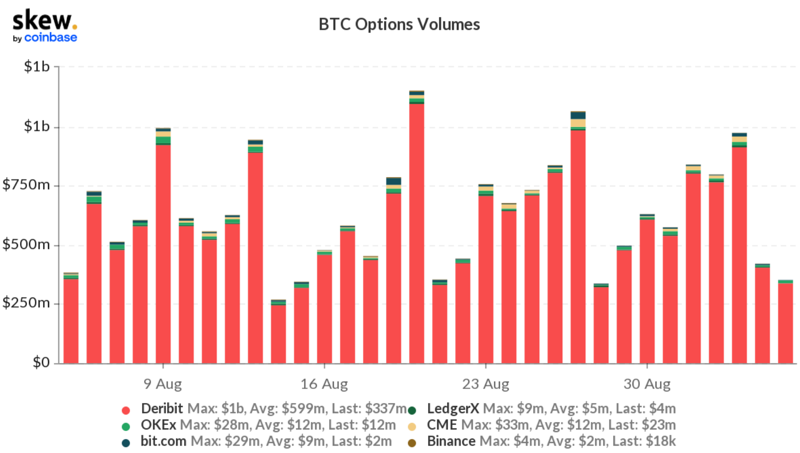
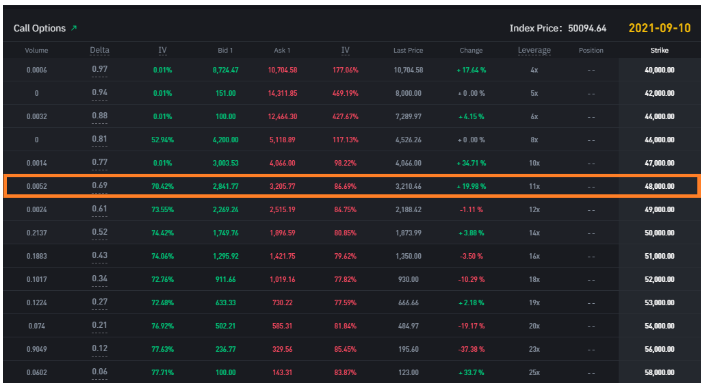


Figure 2.3: Total volume of options transactions for BTC

instruments for other market participants, while providing commitment to buy and sell at the quoted prices. These firms profit from the bid-ask spread to make profit. This distorts the prices and implies that market users aren't trading fair prices.



Call Options ↗ Index Price: 50094.64 2021-09-10

Volume	Delta	IV	Bid 1	Ask 1	IV	Last Price	Change	Leverage	Position	Strike
0.0006	0.97	0.01%	8,724.47	10,704.58	177.06%	10,704.58	+ 17.64 %	4x	--	40,000.00
0	0.94	0.01%	151.00	14,311.85	469.19%	8,000.00	+ 0 .00 %	5x	--	42,000.00
0.0032	0.88	0.01%	100.00	12,464.30	427.67%	7,289.97	+ 4.15 %	6x	--	44,000.00
0	0.81	52.94%	4,200.00	5,118.89	117.13%	4,526.26	+ 0 .00 %	8x	--	46,000.00
0.0014	0.77	0.01%	3,003.53	4,066.00	98.22%	4,066.00	+ 34.71 %	10x	--	47,000.00
0.0052	0.69	70.42%	2,841.77	3,205.77	86.69%	3,210.46	+ 19.98 %	11x	--	48,000.00
0.0024	0.61	73.55%	2,269.24	2,515.19	84.75%	2,188.42	-1.11 %	12x	--	49,000.00
0.2137	0.52	74.42%	1,749.76	1,896.59	80.85%	1,873.99	+ 3.88 %	14x	--	50,000.00
0.1883	0.43	74.06%	1,295.92	1,421.75	79.62%	1,350.00	-3.50 %	16x	--	51,000.00
0.1017	0.34	72.76%	911.66	1,019.16	77.82%	930.00	-10.29 %	18x	--	52,000.00
0.1224	0.27	72.48%	633.33	730.22	77.59%	666.66	+ 2.18 %	19x	--	53,000.00
0.074	0.21	76.92%	502.21	585.31	81.84%	484.97	-19.17 %	20x	--	54,000.00
0.9049	0.12	77.63%	236.77	329.56	85.45%	195.60	-37.38 %	23x	--	56,000.00
0.0602	0.06	77.71%	100.00	143.31	83.87%	123.00	+ 33.7 %	25x	--	58,000.00

Figure 2.4: Bid-ask prices of a call on BTC

To understand the latter idea let us investigate how spread works. Spread has a deep fundamental value. In order to demonstrate this, let us try to answer a question: what is a measurement in finance and how is price measurement performed? If a number represents a valid security price, then there must be parties in the market willing to transact the security at that price. In order to test if the price is right, we must submit an order, for example a BUY order, at a discount price and keep increasing the price until somebody wishes to sell at our price. Once our order executes, we can say that price has been measured and the transaction price represents a valid security price. In fact, every transaction in financial markets is an elementary act of price measurement.
How do we improve price accuracy? If we start with a small order it may not have enough weight to represent price, so we may want to increase order size. It may work to a certain extent. However, at some point the order will become so large that it will affect price.

Other traders will see it and adjust their orders,or our order will execute piercing multiple levels of order book. Even though the spread may still be low, price itself will become distorted. Apparently, there is an inherent price uncertainty associated with the nature of price measurement. That uncertainty cannot be reduced and is directly related to spread. This quality has been pointed out in to resemble the Heisenberg's uncertainty principle in quantum mechanics. For further details see (Sarkissian, 2016).

2.5.3 Transactions fees

The fees for trading bitcoin on Deribit are 7.5bps for Takers and -2.5bps for takers. These fees seem to be insignificant yet when trading at high frequency or even doing short term trades for a long period of time might have a cost on the trading strategy. Below is a chart taken from twitter that shows the performance of trading 1 Bitcoin with executing a buy at 9am and a sell 11pm UTC. Discarding fees at a first time, counting one taker leg fee at a second one and counting both taker fees finally.

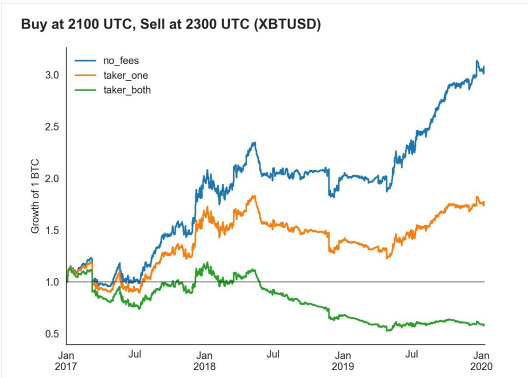


Figure 2.5: Comparaison of trading strategies' performances with or without trading fees

As we can see, executing taker fee on both legs generates significant performance deterioration and turns a profitable strategy into a losing one. This, in a certain way, proves for short term trading transaction fees should be included in the modelling process. Is it the case for Bitcoins' options?

As we can see in the table below there is a difference between taker and maker fees in order to help provide more liquidity on the market. However for bitcoin and Etherium options the fees are equal. Which might be among the reasons why we have lower liquidity in the options market.

Contracts	Maker Fee	Taker fee
BTC Weekly Futures	-0.01% (rebate)	0.05%
BTC/ETH Perps	0.00%	0.05%
BTC/ETH Options	$3bps$ of the underlying asset	$3bps$ of the underlying asset

For Deribit options, the fees can't get higher than 12.5% of the option's value or 3bps or the underlying price. We must say that the fee rarely reaches 12.5% of the option's value. As seen for short term trading, transaction fees have a significant impact on the performance of an investor strategy. However on the long horizon, those fees hardly reach 12.5% and range between 20 and 70bps of the option price. In the modelling process, later on, we choose not to involve transaction fee in order to simplify the modelling process.

2.6 Noise trading

As opposed to traditional markets, many cryptocurrency traders are inexperienced and new to the market. The digital currency price may be therefore dominated by the "noise trader" behavior as described in (Long, Shleifer, Summers & Waldmann, 1990). These investors have no access to inside and technical analysis market's information yet they irrationally act on noise as if it were information that would give them an edge. This is behaviour is spotted every so often on regular yet highly speculative markets like the Chinese stock markets. See (Hou, 2013). Although trading bitcoin exhibits a lot of irrational and psychology based behaviour Bitcoin runs on the blockchain where transactions are recorded and stored in a distributed ledger. This network produces immense amounts of financial data that we can use to gain insights into the health and activity of the network.

2.7 Decentralized Finance

Finally, one must mention one of the very innovative aspects of blockchain: Decentralized markets[12]. First of all let's define Centralized Finance (CeFi) and Decentralized Finance (DeFi). Kraken, Binance, Coinbase ... are well-known CeFi exchanges that have been managing crypto related financial transactions. These exchanges monitor and survey all transactions. The issue is that these platforms are trusted with all users data and money. Hence, there is always a possibility of data security breach since like any other platform they are not immune to data leaks or attacks. Defi, however, have a decentralized structure. Being superbly secure, there is no chance of funds being stolen or misused or vulnerable to thefts. Decentralized Finance is hosted on a Blockchain platform like Ethereum, smart contracts are designed to automatically execute transactions when a particular condition is fulfilled. Since smart contracts are automatic, users can be fully assured that transactions will never fail and will be properly executed. Some of the biggest examples of Decentralized Finance (DeFi) exchanges are Kyber, Totle, MakerDAO etc. There is several advantages for DeFi like the elimination of brokers which results in low trading costs and the accelerated trading process. For cryptocurrency traders or any traders for that matter, even a millisecond delay can lead to huge losses. Appropriate calculation of margins, managing and hedging risks is very essential for any cryptocurrency trading platform. Blockchain

[12] (*Decentralized Finance*, n.d.)

and smart contracts can help in this regard where smart contracts can automate the entire process with proper business rules and logic in place.

2.7.1 Decentralized insurance

Decentralized Insurance[13] is an insurance offer provided for the Decentralized Finance sector.
This offer extends to different products such as :

- Crypto Wallet Insurance: With the growth of the market comes a growing need for crypto wallet insurance solutions. Many companies such as Etherisc have developed protection solutions against crypto wallet theft and hack, covering relatively large sums.
- Collateral Protection for Crypto-backed loans: For Crypto loans, if the collateral provided by the borrower is attacked, destroyed, or stolen, the loan is repaid by the insurer. Several companies like Etherisc Sweetbridge, Celsius, Nexo, and Libra Credit, to cite a few, built an alliance that protects and secures collateralized crypto-backed loans.
- Smart Contract Cover: This insurance policy, developed for instance by Nexus Mutual, covers the potential loss for a designated smart contract. Typically, if the investor account suffers a loss of funds due to its smart contract's address being hacked and manipulated. Or if the funds were transferred to another address, that does not belong to the original investor and can not be recovered. Thanks to these policies, investors can lend crypto loans on the exchange without worrying about losses or repayment

Decentralized insurance is in many ways similar to parametric insurance (automated execution) but offers more personalized solutions.

[13](*Decentralized Insurance — An emerging sector in DeFi*, n.d.)

Chapter 3

Bitcoin market's modelling: A marathon of classic models

"All models are wrong, but some are useful."

George E. P. Box

3.1 Introduction

In an applied mathematics workshop, Christian Mercat, head of IREM Lyon at the time, once told me "Modelling is knowing what is important and what is not". All models by necessity distort reality in one way or another however they help us simulate and predict by simplifying the real world.
This reminds me of a situation described by Mandelbrot and Hudson about an engineer, a physicist, and an economist that find themselves shipwrecked on a desert island with nothing to eat but a sealed can of beans. How to get at them? The engineer proposes breaking the can open with a rock. The physicist suggests heating the can in the sun, until it bursts. The economist's approach: "First, assume we have a can opener. ...".
In finance, economists and financial engineers usually apply very strong assumptions to simplify the mathematics needed to study markets. For instance, as suggests (Kakushadze, 2019), statistical arbitrage might be present in the course of bitcoin yet we choose to go with the no free lunch assumption. Otherwise, The Fundamental theorem of asset pricing can no longer be applied and thus no classical model can be used.
In the scope of the next chapter we will try to price vanilla bitcoin options using some of the most known stochastic models.

3.2 Black & Scholes

3.2.1 Introduction

The story of option pricing began in 1900 when Louis Bachelier, father of quantitative finance and an insufficiently recognized genius, came up with the earliest probabilistic theory known for financial markets as part of his PhD thesis: The theory of Speculation. His happiest thought was that changes in the stock price follow a random walk process. Bachelier revealed the vastness of the world dominated by randomness. Following his dissertation, he came up with a theory of "probabilities". It was the theory of what 30 years later became known as Markovian processes. While Bachelier was on the right track, his model had glaring imperfections. As no discounting was taken into account and prices were allowed to be negative. Yet, it took more than half a decade to propose any alternative models. The Black-Scholes model was then introduced and considered as a revolution in the finance industry as it solved the aforementioned problems.

3.2.2 Definitions and assumptions

Brownian motion

Definition 1 *Let be W an $\mathcal{F}$-adapted process with continuous paths where $\mathcal{F}$ is the natural filtration with $W_0 = 0$. Then, we have that W is a Brownian motion if it verifies the following properties :*

- *W increments are independent* : $W_t - W_x \perp \mathcal{F}_x$
- *W increments are stationary* : $W_t - W_x \sim W_{t-x}$.
- $W_t = \mathcal{N}(0, t)$

Transaction fees

As discussed in 2.1, the transaction fees are ought to be neglected in the modelling process. In order to simplify the equations to solve. However a full analysis should be done to verify the impact of the different costs on long and short term strategies according to their turnover.

Interest rate

For the sake of simplicity the interest rate used for modelling the Crypto currency free risk rate is the USD risk free rate (or treasury rate).

Assumptions

Black-Scholes model assert some strong assumptions:

- Markets are efficient.
- No transaction costs, No transaction volumes restrictions.
- No Arbitrage opportunities.
- The returns of the underlying are Gaussian, stationary and independent:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

- The investment at the bank is risk-free and the interest rate r is constant

3.2.3 Mathematical model

The underlying processes is modeled over a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]} \subseteq \mathcal{F}$ is a filtration satisfying the usual assumptions[1] (F_0 is assumed to be trivial).

We denote by S, the price of risky securities. S is assumed to be càdlàg $\mathbb{F}$-semimartingales.

Let $W^{\mathbb{Q}} = \left(W_t^{\mathbb{Q}}\right)_{t\in[0,T]}$ be a $(\mathbb{F})$-Brownian motion. We introduce the coefficient functions $\mu : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}, \sigma : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$, which are assumed to satisfy standard conditions ensuring existence and uniqueness of strong solutions of the following SDE:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \text{ for } \mu, \sigma > 0 \tag{3.1}$$

In the Black-Scholes frame μ and σ are set to be constants. Using the Itô formula to $v(t, S_t)$ we can deduce the following equation:

$$dv(t, S_t) = \left(\frac{\partial v}{\partial t} + \mu S_t \frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}\right) dt + \sigma S_t \frac{\partial v}{\partial S} dW_t$$

Assume the following portfolio $V = v(t, S_t) - \frac{\partial v}{\partial S} S_t$. Applying the itô formula we get

$$dV = dv - \frac{\partial v}{\partial S} dS = \left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S^2}\right) dt$$

Therefore V is a non risky portfolio and we get that $\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2}\right) dt = rV dt$ from which we have the following theorem:

[1] A filtered probability space is said to satisfy the usual conditions if it is complete (i.e., $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and right-continuous (i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all times t)

Theorem 1 *(Black-Scholes PDE). In the Black-Scholes model, the price at date t of a European option with pay-off $g(S_T)$ at date T is given by $v(t, S_t)$, where the function v is the solution on $[0,T] \times (0,\infty)$ of the PDE*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial S^2}\sigma^2 S_t^2 = r\left(v - S_t\frac{\partial v}{\partial S}\right), \quad v(T,S) = g(S) \tag{3.2}$$

Using the Feynman-Kac theorem and the Markov property of the process one can see that:

Theorem 2 *Let g be a polynomial growing function, i.e., there exists p such that $|g(x)| \leq C(1+|x|^p)$ for all x. Then the PDE 3.2 admits the unique solution in the class of polynomial growing functions, belonging to $C^0([0,T] \times (0,\infty)) \cap C^{1,2}([0,T] \times (0,\infty))$ given by*

$$v(t,S) = \mathbb{E}\left[e^{-r(T-t)}g\left(Se^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma W_{T-t}}\right)\right]$$

Finally the pricing formula for a call option is:

Theorem 3 *(Black-Scholes formula). The price of a European call option with payoff $g(S) = (S-K)^+$ in the Black-Scholes model is given by*

$$v(t,S) := C_{BS}(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{3.3}$$

where

$$d_{1,2} = \frac{log\frac{S}{Ke^{-r(T-t)}} \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and N is the distribution function of the normal distribution:

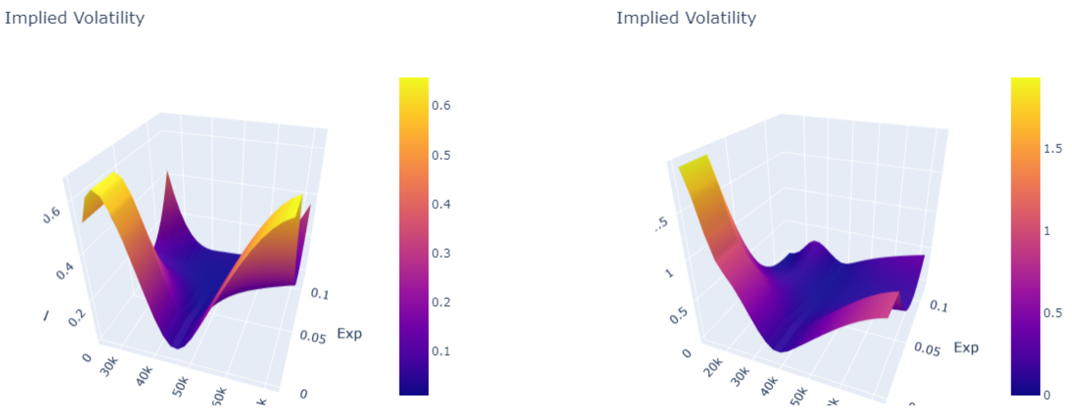
$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}dz$$

3.2.4 Volatility surface

In the scope of this section we investigate the implied volatility of bitcoin's options. Volatility is a measure of the level of fluctuations in a certain price. How we measure it, what unit we use, what time scale we are looking at, all of these matters have an impact and have to be clarified to make a single volatility figure a robust comprehension of how much that particular stock is fluctuating. Implied volatility is a more market-related concept of volatility. Taken the market price, implied volatility ensures that the Black-Scholes formula derived price matches the observed price. The function, $\sigma \mapsto C_{BS}(\sigma)$ which associates the corresponding Black-Scholes price to a volatility value, and verifies:

$$\lim_{\sigma\downarrow 0} C_{BS}(\sigma) = \left(S - Ke^{-r(T-t)}\right)^+ \quad \lim_{\sigma\uparrow\infty} C_{BS}(\sigma) = S$$

$$\frac{\partial C_{BS}}{\partial \sigma} = Sn(d_1)\sqrt{T-t} > 0$$



(a) Implied volatility surface on 17/08/2021 (b) Implied volatility surface on 30/07/2021

Figure 3.1: Implied volatility surface (Interpolation method is cubic spline)

This implies that the equation $C_{BS}(\sigma) = C$ has a unique solution for any value of C satisfying the arbitrage constraints:

$$\left(S - Ke^{-r(T-t)}\right)^{+} < C < S$$

Results

To solve the latter inversion problem two numerical methods were used: The Newton-Ralphson and bisection methods. Yet Newton-Ralphson algorithm was faster to converge as it needed no more than 5 to 6 iterations. One can see the different surfaces interpolated using cubic spline method in fig 3.1.

Interpretation

The interpretation of the volatility surface for bitcoin's derivative market is a little bit tricky. We will try to check if we can observe the same stylized facts seen in traditional markets.
As can be seen in the figures above, the volatility surface for BTC options isn't constant both smiles and skews can be spotted.
Asking the question of what causes volatility smile or skew can help us understand the particularity of this surface:
The smile is due to several reasons. The most important one is probably supply and demand. For instance in the FX market the smile is symmetrical. For example, taking the case of EUR/USD. Euro investors see the market the opposite way dollar investors observe it. We can see a shade of this symmetrical behaviour in 3.1 on the 17^{th} of august. However, for bitcoin the reason might be a little bit different than that. Since many investors

use bitcoin options equally for hedging risk and for speculation, they might sometimes be interested in both ITM[2] calls and OTM[3] puts.

Looking at the same figure but on the 30^{th} of July, the volatility surface exhibits a heavy skew just like the equity market and that is due to the fact that investors have to protect themselves against large drops specifically buying more protective puts to hedge that risk. This actually makes sense since Bitcoin price knew a huge drop in the end of July.

For the second data set. The forward volatility skew is fairly apparent. The forward volatility skew is just a reverse form of the usually observed volatility smirk. To have a clearer vision of the forward skew, it is a certain form of the surface where OTM calls and ITM puts are priced at a much higher implied volatility. This forward skew observed in Bitcoin options suggests that the demand for buying out-of-the-money calls and in-the-money puts has increased significantly not only to hedge the Bitcoin price risk yet to speculate its increase.

Eventually, as the options approach maturity, the implied volatilities rose to more than 180% for some trading day. Therefore, the appropriate interpretation of the increase in implied volatilities is the demand for these strikes. An additional potential reason for this forward volatility skew is a notable buying interest in OTM calls, as several investment institutions are interested in offering their custom products in exchange for cryptos. The increased interest of institutional investors and cryptocurrency practitioners, among many other factors, drives upward the Bitcoin price which eventually increases the implied volatility of options with high strikes.

When we move in time and the curve evolves, we can see that the forward skew became more symmetrical and pronounced several days before expiration. The volatility smile is the deepest for short-term options near expiration. Movements like these are of tremendous importance to both speculators and investors, since I believe they suggest that speculators are more willing to plunge into the Bitcoin market when a volatility smile emerges close to expiry. In fact, as speculative trades mostly are close to expiry, an increasing interest is seen for ITM and OTM options rather than ATM ones. This demand rise yet supply shortage increases the extrinsic value of options increasing therefore the implied volatility. This phenomenon is seen in the volatile Bitcoin and cryptocurrency market, as implied volatility can go up from 60 % to a staggering 200 % in a short period of time (14 trading days).

By looking at the properties of the bitcoin volatility smile, the existence of forward volatility skewness resembles the skewness of traditional commodity markets rather than equity ones. One can conclude that Bitcoin might belong to the commodity class of assets. This conclusion is actually backed by the Commodity Futures Trading Commission that

[2] In the money: when the execution price is higher than the price of the underlying asset

[3] Out of the money: when the exercise price is lower than the price of the underlying asset

announced that Bitcoin can work as a commodity like gold, silver... and thus can work as portfolio diversifier and a hedging option as it is not very correlated with other equity assets.

Conclusion The Black-Scholes model can reveal some properties about Bitcoin's price: First of all, this model is not appropriate to capture BTC dynamics as we can see that the implied volatility surface is not flat.
In addition, the changing surface, suggest the use of a model where the volatility acts upon a certain dynamics. These two findings, motivate the use of two more complex model like a jump diffusion and a stochastic volatility model. In section 3 and section 4 we are going to fit the Merton and Heston stochastic volatility model respectively to see whether they succeed in capturing the Bitcoin's price characteristics.

3.3 Merton model

3.3.1 Introduction

The Black-scholes model was a mixed blessing. On the one hand, the model was superbly simple and used to price a lot of financial products. On the other one, it failed capturing the volatility smile and skew. In an attempt to solve this problem (Merton, 1976) in his paper "Option Pricing When Underlying Stock Returns Are discontinuous" and (Cox & Ross, 1976) allowed the stock to exhibit jumps. Transforming the process from a simple continuous diffusion process to a jump diffusion one. The purpose of his paper was to make the Black-Scholes model more realistic and able to deal with the fact that empirical studies of market returns show that returns do not follow a constant variance log-normal distribution. The Merton model, properly calibrated to the market data, has been successful in producing skews in volatility. For example, short-term skews can be captured when introducing a negative mean for the jump process.

3.3.2 The Mathematical model

Poisson process

Definition 2 *A poisson process with intensity λ is an occurrence counting process (non-negative, integer-valued, increasing cadlag stochastic process) that verifies these conditions:*
1. $\forall t_0 = 0 \leq t_1 \leq ... \leq t_k$ the variables $(N_{t_k} - N_{t_{k-1}}), ...(N_{t_1} - N_{t_0})$ are independants.
2. $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$ when $h \to 0^+$.
3. $\mathbb{P}(N_{t+h} - N_t > 1) = o(h)$ when $h \to 0^+$
therefore $P(N_{t+h} - N_t = 0) = 1 - \lambda h + o(h)$ when $h \to 0^+$.

Compound poisson process

Definition 3 *A compound poisson process is defined as:*

$$Q_t = \sum_{i=1}^{N_t} J_i \tag{3.4}$$

The process N_t is a Poisson process and the sequence of random variables J_i are assumed to be independent. A jump at time t is therefore $J_i * dN_t$ Where dN_t is a Bernouilli random variable according to 2

Assumptions For this model we use:

- The same interest rate as the Black-Scholes model.
- We assume that there is no arbitrage opportunities.

- No transaction costs, No transaction volumes restrictions
- The jumps to be independent from the diffusion process.
- That the jumps are independent from one another.

The mathematical model

The dynamics of the price process is given by:

$$\begin{cases} dX_t = \mu dt + \sigma dW_t + J dN_t \\ h(X_t) = \exp(X_t) = S_t \end{cases}$$

where σ and μ are constants and the jump sizes are normal $\forall i$, $J_i \sim \mathcal{N}(m, \delta^2)$. Therefore let be $Q_t = \sum_{i=1}^{N_t} J_i$, a compound Poisson process describing the jump process. Following (El-Himdi, 2020) work at *Mazars Actuariat* and using the Itô formula for jump and diffusion processes. We find that:

$$dS_t = \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t + S_{t^-}(e^{J_i} - 1) dN_t$$

Let be $M_t = \sum_{i=1}^{N_t} \left(e^{J_i} - 1\right) - \lambda \mathbb{E}\left(e^{J_i} - 1\right) t$ is a martingale representing the centred jump process (λt is the expectation of the Poisson process and λ is the jumps occurrence intensity). Therefore:

$$dS_t = \left(\mu + \frac{\sigma^2}{2} + \lambda k\right) S_t dt + \sigma S_t dW_t + S_{t^-} dM_t \tag{3.5}$$

Where $k = e^{m + \frac{\delta^2}{2}} - 1$ and. Using the risk-neutral probability $\mathbb{Q}$ Eq. 3.5 becomes

$$dS_t = r S_t dt + \sigma S_t dW_t^{\mathbb{Q}} + S_{t^-} dM_t^{\mathbb{Q}}$$

where $W_t^{\mathbb{Q}} = W_t + \theta_t$, and $\theta_t = \frac{\mu + \sigma^2/2 + \lambda k - r}{\sigma}$ Thus,

$$dS_t = (r - \lambda k) S_t dt + \sigma S_t dW_t^{\mathbb{Q}} + S_{t^-}(Y - 1) dN_t$$

and the price process is therefore

$$S_t = S_0 \exp\left(\left(r - \sigma^2/2 - \lambda k\right) t + \sigma dW_t^{\mathbb{Q}} + \sum_{i=1}^{N_t} J_i\right)$$

Using Markov property, we can deduce the option price:

$$
\begin{aligned}
c(t,T,K,S_t) &= e^{-r(T-t)}\mathbb{E}\left[\left((S_T-K)_+\right) \mid S_t = S\right] \\
&= e^{-r(T-t)}\mathbb{E}\left[\psi\left(Se^{\left((r-\frac{\sigma^2}{2}-\lambda k)(T-t)+\sigma W^{\mathbb{Q}}_{T-t}+\sum_1^{N_{T-t}} J_i\right)}\right)\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty}\mathbb{Q}(N_t=n)\,\mathbb{E}\left[\psi\left(Se^{(r-\frac{\sigma^2}{2}-\lambda k)(T-t)+\sigma W^{\mathbb{Q}}_{T-t}+\sum_1^{n} J_i}\right)\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty}e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}\mathbb{E}\left[\psi\left(Se^{(r-\frac{\sigma^2}{2}-\lambda k)(T-t)+nm+\sqrt{\sigma^2+\frac{n\delta^2}{t}}W^{\mathbb{Q}}_{T-t}}\right)\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty}e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}\mathbb{E}\left[\psi\left(Se^{(r-\frac{\sigma_n^2}{2})\tau-\lambda k\tau+\frac{n\delta^2}{2t}\tau+nm+\sqrt{\sigma^2+\frac{n\delta^2}{t}}W^{\mathbb{Q}}_{\tau}}\right)\right]
\end{aligned}
$$

Therefore the price of a call option is:

$$
c(t,T,K,S_t) = e^{-r(T-t)}\sum_{n=0}^{\infty}e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}c_{bs}(\tau, S_{n,t}, \sigma_n)
$$

where ψ is an integrable measurable function, $\tau = T-t, S_{n,t} = Se^{nm+n\delta^2/2t-\lambda k\tau}, \sigma_n^2 = \sigma^2 + \frac{n\delta^2}{t}$ and c_{bs} is the BS call option price.

3.3.3 Model calibration

In order for the model prices to fit the market, its parameters should be calibrated. Among the advantages of this model, is that it gives us closed formulas that can be relatively easy to calibrate. The optimization problem that we have to solve is the following:

$$
\min_{\boldsymbol{\theta}\in\mathbb{R}^b} f(\boldsymbol{\theta}) \tag{3.6}
$$

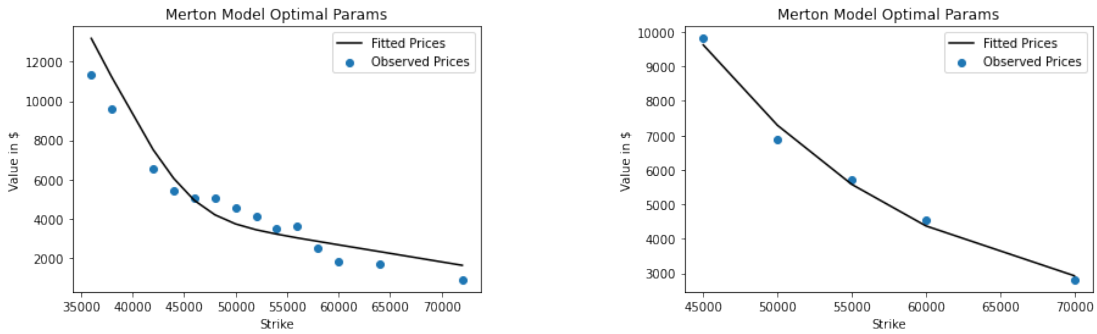
where $f(\boldsymbol{\theta}) := \frac{1}{2}\|\boldsymbol{r}(\boldsymbol{\theta})\|^2 = \frac{1}{2}\boldsymbol{r}^\top(\boldsymbol{\theta})\boldsymbol{r}(\boldsymbol{\theta})$ and $\boldsymbol{r}(\boldsymbol{\theta}) := [r_1(\boldsymbol{\theta}),\ldots,r_n(\boldsymbol{\theta})]^\top$ and finally

$$
r_i(\boldsymbol{\theta}) := c(\boldsymbol{\theta}; K_i, T_i) - c^*(K_i, T_i), \quad i = 1,\ldots,n
$$

We have that C^* the market price and C the model price depending on the parameters' vector $\boldsymbol{\theta} := [\lambda,\sigma,\delta,m]^\top$ of size $b = 4$. And given the following bounds:

$$
\begin{gathered}
0 < \sigma < \infty \\
0 < m < 2 \\
0 < \delta < \infty \\
0 \leq \lambda < 10
\end{gathered}
$$

The algorithm used to solve this problem is the Sequential Least Squared Quadratic Programming (SLSQP) developed by (Kraft, 1988) already implemented in *Python*'s package `scipy`



(a) Price slice short maturity (b) Price slice long maturity

Figure 3.2: Price slices on the 17th August

Results

As we can see the Merton jump diffusion model was able to price the vanilla options with a relative mean squared error of 14.89 %. The results of the calibration process yielded these parameters.

λ	σ	δ^2	m
5.020	0.607	0.042	1.914

When compared to the Black-Scholes model The Merton model was able to reproduce in a way the volatility smile as we see in 3.3. The strength of jump diffusion processes is that it fits short-term skews. The skew for a given stock is greater for short-term maturities than for long-term maturities. For long maturities, the skew might have a higher level, but the skew in the short term options is more pronounced. As a matter of fact, the skew is present because traders and investors are generally concerned about losing money in case the market becomes volatile and goes down. Finally, one should mention that the larger Γ the more problematic it is. Short-term options with low strikes have larger Γs so whenever the underlying price decreases, the skew is steeper. And, here comes the effect of adding jumps in the underlying's dynamics, as a jump would have a tremendous impact on the price of an option because for the short term the market may not have time to be back to normal from a certain sudden variation.

Conclusion

The Merton model was able to reproduce in a way the volatility smile. Using a jump process helped with fitting short-term skews. The skew for a given stock is greater for short-term maturities than for long-term maturities. However, clearly, the Merton model cannot capture the changing dynamics of the volatility process. In the following section, we take advantage of the Heston stochastic volatility model.

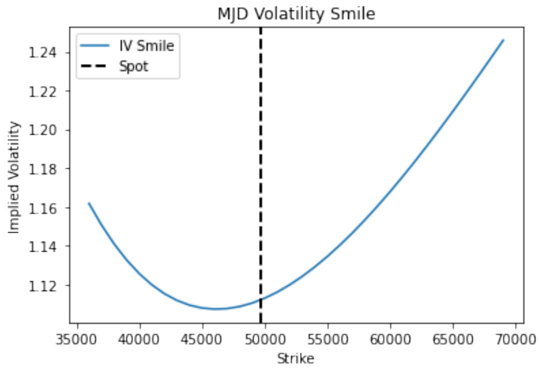


Figure 3.3: Volatility smile

Remark As a matter of fact, since Deribit and most other exchanges only offer vanilla options, we could have used a local volatility model that will be able to, perfectly reproduce the volatility surface. However, we chose to use parametric models in order for us to better understand the properties of the Bitcoin derivatives' market.

3.4 Stochastic volatility

3.4.1 Introduction

As discussed in the latter section a stochastic volatility model is needed to capture more information about the cryptocurrency dynamics that cannot be captured by other models. One can also see from 3.1 that the volatility surface changes dynamically which arises a risk that should be captured by the model. To do so, we chose the Heston model as it is considered to be the bedrock on which lies all stochastic volatilises models. This model provides a closed-form solution for European Call options, that can be very useful in the calibration process. It also allows the underlying price process to follow a non log-Normal probability distribution and ensures the mean-reverting stylized fact of the volatility process therefore fits fairly well the market implied volatility surface. Finally, we can put into account the correlation of price and volatility using this model.

3.4.2 The mathematical model

Assumptions

For this model we use:

- The same interest rate as the Black-Scholes model.
- We assume that there is no arbitrage opportunities.
- No transaction costs, No transaction volumes restrictions
- The volatility follows a mean reversion process
- Both the risk factor of volatility and price process are constantly correlated.

The model

The underlying processes (S_t, V_t) is modeled over a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]} \subseteq \mathcal{F}$ is a filtration satisfying the usual assumptions.
Let S_t be the the price process and V_t be the volatility process. The dynamics of the Heston model are described as follows.

$$\begin{gathered} dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(S)} \\ dV_t = \kappa\left(\bar{v} - V_t\right) dt + \sigma\sqrt{V_t} dW_t^{(V)} \\ d\left\langle W_t^{(S)}, W_t^{(V)}\right\rangle = \rho dt \end{gathered} \tag{3.7}$$

Looking at the equations above we can spot that the diffusion coefficient is positive (square root of an entity) which results in the non negativity of the variance. When invetigating historical data, we can easily see that volatility is not static and it varies over time. Volatility actually seems to oscillate around a long-term mean level. That's why it is modeled using a square root mean-reverting process. A shadow of this can be seen in interest rates' term structure modelling by Cox, Ingersoll and Ross. See the T3Index's bitvol which is made to resemble the VIX index of the S&P 500 fig 3.4

Figure 3.4: Bitcoin volatility index, source: T3Index.

The drift term in the volatility dynamics is mean reverting if $\kappa > 0$, with $\bar{v}$ being the long-term mean level of the variance. In fact, if at time t the process V_t is greater than θ, the drift term will push the process value down.

3.4.3 Calibration of the model

The calibration of the Heston model is trickier than the other models. First of all there are 5 paramaters that need to be calibrated and the objective function is not known to be convex. A complexity is added later on due to the dependency of some parameters. As a matter of fact σ and κ offset each other. That's might be due to the fact that the objective function is flat reaching the optimum.
Like the other models we will try to minimise the spread between a vanilla option's price and the one found on the market. The objective function is no different then other classic optimization problems:

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^m} f(\boldsymbol{\theta}) \tag{3.8}$$

where $f(\boldsymbol{\theta}) := \frac{1}{2}\|\boldsymbol{r}(\boldsymbol{\theta})\|^2 = \frac{1}{2}\boldsymbol{r}^\top(\boldsymbol{\theta})\boldsymbol{r}(\boldsymbol{\theta})$ and $\boldsymbol{r}(\boldsymbol{\theta}) := [r_1(\boldsymbol{\theta}), \ldots, r_n(\boldsymbol{\theta})]^\top$ and finally

$$r_i(\boldsymbol{\theta}) := C(\boldsymbol{\theta}; K_i, T_i) - C^*(K_i, T_i), \quad i = 1, \ldots, n$$

We have that C^* the market price and C the model price depending on the parameters' vector $\boldsymbol{\theta} := [v_0, \bar{v}, \rho, \kappa, \sigma]^\top$ of size $m = 5$. Since the, the explicit gradient of C with respect to θ is deemed to be overly complicated. We will use the (del Baño Rollin, Ferreiro-Castilla & Utzet, 2010) formulation used by (Cui, del Baño Rollin & Germano, 2016) in his calibration process of the Heston process.

The pricing formula of a call in the Heston model was given by the original auther as:

$$C(\boldsymbol{\theta};K,T)=\tfrac{1}{2}\left(S_0-e^{-rT}K\right)+\tfrac{e^{-rT}}{\pi}\left[\int_0^\infty \mathrm{Re}\left(\tfrac{e^{-iu\log K}}{iu}\phi(\boldsymbol{\theta};u-i,T)\right)\mathrm{d}u \right.$$
$$\left. -K\int_0^\infty \mathrm{Re}\left(\tfrac{e^{-iu\log K}}{iu}\phi(\boldsymbol{\theta};u,T)\right)\mathrm{d}u\right]$$

Where ϕ is the logarithmic price characteristic function.

$$\phi(\boldsymbol{\theta};u,t)=\exp\left\{iu\left(\log S_0+rt\right)+\tfrac{\kappa\bar{v}}{\sigma^2}\left[(\xi+d)t-2\log\tfrac{1-g_1e^{dt}}{1-g_1}\right] \right.$$
$$\left. +\tfrac{v_0}{\sigma^2}(\xi+d)\tfrac{1-e^{dt}}{1-g_1e^{dt}}\right\}$$

Where

$$\xi:=\kappa-\sigma\rho iu$$
$$d:=\sqrt{\xi^2+\sigma^2\left(u^2+iu\right)}$$
$$g_1:=\frac{\xi+d}{\xi-d}$$

The problem of this original formulation is that for relatively long maturities discontinuities appear because of the branch switching of the complex power function which appears in the characteristic function. See (Cui et al., 2016) for further details about this problem. To calibrate the model we use their characteristic function formulation as they proved it solved the discontinuity problem rather than that its analytical gradient is easy to compute.

$$\phi(\boldsymbol{\theta};u,t)=\exp\left\{iu\left(\log S_0+rt\right)-\frac{t\kappa\bar{v}\rho iu}{\sigma}-v_0A+\frac{2\kappa\bar{v}}{\sigma^2}D\right\} \tag{3.9}$$

Where,

$$D=\log d+\frac{\kappa t}{2}-\log A_2$$
$$=\log d+\frac{(\kappa-d)t}{2}-\log\left(\frac{d+\xi}{2}+\frac{d-\xi}{2}e^{-dt}\right)=:logB$$

and

$$\log A_2=\log\left(d\cosh\frac{dt}{2}+\xi\sinh\frac{dt}{2}\right)\quad=\frac{dt}{2}+\log\left(\frac{d+\xi}{2}+\frac{d-\xi}{2}e^{-dt}\right)$$

The gradient of the price function is:

$$\boldsymbol{\nabla}C(\boldsymbol{\theta};K,T)=\tfrac{e^{-rT}}{\pi}\left[\int_0^\infty \mathrm{Re}\left(\tfrac{K^{-iu}}{iu}\boldsymbol{\nabla}\phi(\boldsymbol{\theta};u-i,T)\right)\mathrm{d}u\right.$$
$$\left.-K\int_0^\infty \mathrm{Re}\left(\tfrac{K^{-iu}}{iu}\boldsymbol{\nabla}\phi(\boldsymbol{\theta};u,T)\right)\mathrm{d}u\right]$$

where $\boldsymbol{\nabla}\phi(\boldsymbol{\theta}; u, T) = \phi(\boldsymbol{\theta}; u, T)\boldsymbol{h}(u), \boldsymbol{h}(u) := [h_1(u), h_2(u), \ldots, h_5(u)]^\top$ with elements

$$\begin{aligned}
h_1(u) &= -A \\
h_2(u) &= \frac{2\kappa}{\sigma^2}D - \frac{t\kappa\rho iu}{\sigma} \\
h_3(u) &= -v_0\frac{\partial A}{\partial\rho} + \frac{2\kappa\bar{v}}{\sigma^2 d}\left(\frac{\partial d}{\partial\rho} - \frac{d}{A_2}\frac{\partial A_2}{\partial\rho}\right) - \frac{t\kappa\bar{v}iu}{\sigma} \\
h_4(u) &= \frac{v_0}{\sigma iu}\frac{\partial A}{\partial\rho} + \frac{2\bar{v}}{\sigma^2}D + \frac{2\kappa\bar{v}}{\sigma^2 B}\frac{\partial B}{\partial\kappa} - \frac{t\bar{v}\rho iu}{\sigma} \\
h_5(u) &= -v_0\frac{\partial A}{\partial\sigma} - \frac{4\kappa\bar{v}}{\sigma^3}D + \frac{2\kappa\bar{v}}{\sigma^2 d}\left(\frac{\partial d}{\partial\sigma} - \frac{d}{A_2}\frac{\partial A_2}{\partial\sigma}\right) + \frac{t\kappa\bar{v}\rho iu}{\sigma^2}
\end{aligned}$$

To make the calibration process more efficient, (Cui et al., 2016) tried to optimise the computation of the integrands in the equation 3.9. Due to the fact that functions of the form $\mathrm{Re}\left(\phi(\boldsymbol{\theta}; u, t)K^{-iu}/(iu)\right)$ are smooth and decay rapidly: The more spread out the function is the more localised its Fourier transformation (The uncertainty principle). As a matter of fact when T increases S_T probability function widens out and therefore its Fourier transformation squeezes. Based on this property, one can adjust the truncation according to the maturity of the option and hence do fewer integrand evaluations for options with longer maturities.
The integration scheme used in the computation is the Gauss-Legendre as (Cui et al., 2016) showed it's absolute efficiency compared to other schemes.

Finally, to solve the optimization problem we used the Levenberg-Marquardt. The LM method is a typical tool to solve a nonlinear least squares problem, The search step is given by:

$$\Delta\boldsymbol{\theta} = \left(\boldsymbol{J}\boldsymbol{J}^\top + \mu\boldsymbol{I}\right)^{-1}\boldsymbol{\nabla}f$$

By adjusting μ, the method changes between the gradient descent method and the Gauss-Newton method: when the iterate is far from the optimum, μ is given a large value so that the Hessian matrix is dominated by the scaled identity matrix $\nabla\nabla^\top f \approx \mu\boldsymbol{I}$ and when we are close to the optimum μ is small enough that Hessian matrix is dominated by the Gauss-Newton algorithm $\nabla\nabla^\top f \approx JJ^\top$.
The algorithm shall stop iterating as soon as $\|\boldsymbol{r}(\boldsymbol{\theta}_k)\| \leq \varepsilon_1$, $\|\boldsymbol{J}_k\boldsymbol{e}\|_\infty \leq \varepsilon_2$ or $\frac{\|\Delta\boldsymbol{\theta}_k\|}{\|\boldsymbol{\theta}_k\|} \leq \varepsilon_3$ is satisfied.

3.4.4 Results

Ability of the model to capture forward skew When changing the model correlation's parameter we can see that the density skew is captured. Good thing is the Heston model gives us the ability to capture a forward skew by picking a positive correlation between the price and the volatility. As we have seen in fig 3.1 the implied volatility

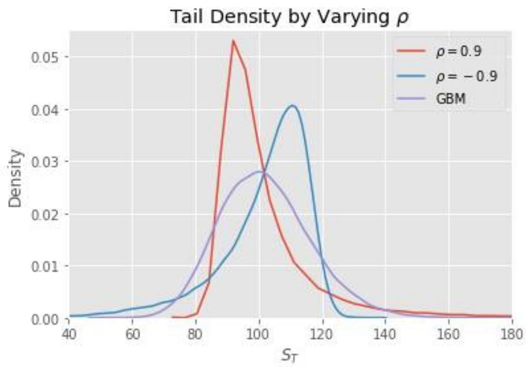


Figure 3.5: Effect of the correlation between price and volatility

surface exhibits a forward skew[4] just like in the commodity market.

3.4.5 Calibration results

The calibration process yields the following parameters

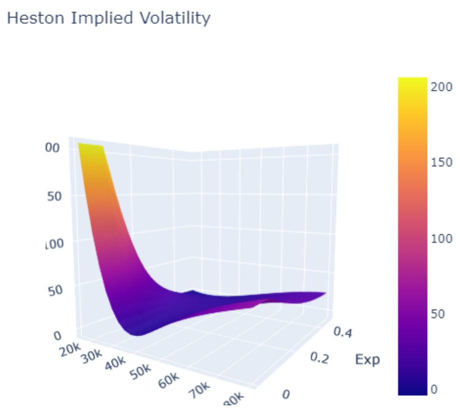
Date	V_0	$\bar{v}$	ρ	κ	σ
30/07/2021	2.8686	1.414	0.7848	3.52158	1.94371
18/08/2021	2.3031	1.524	0.31635	2.1849	1.7041
28/09/2021	0.6917	0.7896	0.08932	0.8674	0.4254

Interpretation

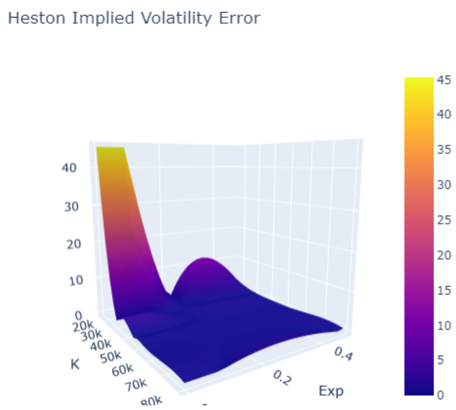
We can see that an increase of $\bar{v}$ corresponds to an increase of the variance and consequently an upward translation of the volatility smile. σ is the volatility of variance σ_v and it controls the kurtosis of the distribution and thus affects the implied volatility. The Heston model has the ability to imitate the volatility smile observed in the market: A larger σ, therefore, implies a more pronounced smile. To better explain this, when the volatility of volatility rises the probability of extreme price changes increases and increases the OTM options price. We can see this feature in fig 3.1.
κ can be seen as the speed of the mean reversion of the volatility. In other words, it indicates the degree of volatility clustering.

It might be seem that the impact of the mean-reversion speed on option prices is restricted. However, its effect appears to be different when options are deeply ITM or deeply OTM. This is closely related to the fact that increasing the speed of the mean reversion of the variance decreases the probability of extreme movements. As a result, when an option is deeply OTM, increasing κ decreases the probability that the option will finish ITM at maturity, therefore decreasing its price. By the same way, when an option is deeply ITM, increasing κ increases the likelihood of the option finishing ITM, therefore increasing its

[4]See appendix for further details on forward skew



(a) Heston Implied volatility surface



(b) Heston Implied volatility surface error

Figure 3.6: Heston Implied volatility surface (on 30/07/2021)

price.

Finally looking at the correlation between stock price and volatility, we can see that, as opposed to many empirical studies that have documented a negative correlation between the stock price and volatility processes (Black, 1976). In the case of bitcoin the correlation parameter ρ is positive. Taking the example of equity markets, when prices of a certain stock declines the leverage that firms have is usually inflated. This results in more uncertainty about the market and therefore a higher volatility. The latter phenomenon is controlled by the correlation parameter in the Heston model. As showed in 3.4.4 this impacts the the skewness of return distribution. For instance a negative correlation means a relatively thicker left tail. Accordingly, the skewness affects the shape of the volatility surface and adds a downward slopinng curve meaning a higher price of deep OTM puts. For the case of bitcoin and some crypto currencies a higher deep OTM calls used for speculation. Hence, the forward skew and the positive correlation parameter.

3.4.6 Pricing results

We used the model to price our options, overall the Heston model performed better than the ones before it. The relative mean squared error of the stochastic volatility model is 11.84% for July the 30^{th} and 9.76% for August the 18^{th}.
Using the Heston prices we derived the implied volatility surface in fig 3.6. First of all we can see that the stochastic volatility model reproduces the volatility smile. When plotting the volatility surface error we notice that the error comes from short term options. Stochastic volatility models also have limitations as they have difficulties in matching the short and long maturity skew or both ends of the volatility surface at the same time.

To solve this problem one can add jumps. As we have seen in section 8.3.2 jumps are fairly capable of the explaining the short term skew. As a matter of fact, the explanation of the strong short-term skew is related to the jumps. However when adding jumps, we don't generally affect the long-term skews that stays relatively flat. We can actually conclude that the long-term skew is not driven by the stock price jumps.

3.4.7 The use of stochastic volatility models

We might not need stochastic volatility models full potential with only vanilla options available in the market right now but with the flourishing market of cryptocurrency more complex derivatives will be traded in the future on different exchanges. Stochastic volatility models, actually, go beyond modelling the skew to allowing the vega convexity and forward skew feature. Therefore, they are capable to price products that exhibit vega convexity or forward skew. These contracts are sensitive to volatility is a non-linear way, meaning that their volga greek (the second-order derivative to volatility) is not null. We can also see a shadow of this feature in vanilla options as they are also convex in volatility, especially OTM options. Yet a Stochastic volatility model is not needed as the risk of vega convexity can be hedged by hedging the skew. Since they are liquidly traded, a Black-Scholes model with market implied volatility should be enough to fairly price them. The way the Heston model, for example, captures Vega convexity in complex payoffs is by the sigma parameter which represents the volatility of volatility. Finally, a really important advantage of this model's type is that it can generate forward skews and therefore efficient in pricing derivatives exhibiting this feature like forward starting and cliquet options which are really popular in commodities market and therefore might be launched later on for cryptocurrency. The way these models do this is by capturing the smile dynamics[5] since the volatility has its own stochastic dynamics.

Conclusion

The stochastic volatility model reproduces the volatility smile. However, the error of this surface comes from short maturities. The Heston model have limitations as they have difficulties in matching the short and long maturity skew or both ends of the volatility surface at the same time. The next step is to add both jumps and stochastic volatility to our model. Therefore we use a stochastic volatility with correlated jumps in the next section.

[5]The phenomena of how the skew changes as the the stock prices changes

3.5 Stochastic volatility with co-jumps

Introduction In the scope of this part we investigate an even more flexible model. Unlike the stochastic volatility model this one can incorporate some of the irregularity of BTC related to jumps seen in its price chart. We decided to get use of both Merton's and Heston's strengths by implementing the stochastic volatility with correlated jumps model of Duffie, Pan and Singleton that adds correlated jumps to both the price and volatility process.

3.5.1 The mathematical model

Assumptions

For this model we use:

- The interest rate is constant.
- We assume that there is no arbitrage opportunities.
- No transaction costs, No transaction volumes restrictions
- Both diffusion are correlated.
- That the jumps are independent from one another and come at a constant rate.
- Jumps generated in the price and volatility processes are correlated.

The model

The underlying processes (S_t, V_t, N_t) is modeled over a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]} \subseteq \mathcal{F}$ is a filtration satisfying the usual assumptions.
Let S_t be the price process and V_t the volatility one, the SVCJ dynamics are as follows:

$$\begin{aligned} d\log S_t &= \mu dt + \sqrt{V_t} dW_t^{(s)} + Z_t^y dN_t \\ dV_t &= \kappa\left(\bar{v} - V_t\right) dt + \sigma\sqrt{V_t} dW_t^{(V)} + Z_t^v dN_t \\ d\left\langle W_t^{(S)}, W_t^{(V)}\right\rangle &= \rho dt \\ \mathbb{P}\left(dN_t = 1\right) &= \lambda dt \end{aligned} \tag{3.10}$$

Like the Heston model, $\kappa, \bar{v}$ denote the mean reversion rate and the mean reversion level, σ is the volvol capturing the variance responsiveness to diffusive volatility in shocks and ρ is the correlation between the diffusion of the price and volatility used to capture the leverage effect.
N_t is a counting process with a constant mean arrival rate λ. The random jump sizes are Z_t^v, Z_t^y. The random jump size Z_t^y conditional on Z_t^v is

$$Z_t^y \mid Z_t^v \sim \mathrm{N}\left(\mu_y + \rho_j Z_t^v, \sigma_y^2\right); \quad Z_t^v \sim \exp\left(\mu_v\right)$$

3.5.2 Calibration of the model

MCMC[6] methods

It has been proven that MCMC methods are of great use when it comes to calibrating SV[7] models. This is not just because Markov chain based algorithms are efficient in terms of computation yet they are flexible methods and therefore can estimate with high precision jump sizes and times and volatility.

MCMC methods that can be applied in more general cases like nonlinear and non-Gaussian models. These algorithms' perk is that it generates the distribution of both state variables and parameters knowing apriori data. This allows for a wide class of numerical fitting procedures that can be guided by a variation of the priors. Among the fundamental theorems used is the Clifford Hammersley theorem. It states that a joint distribution can be characterized by its complete conditional distributions:

$$\mathcal{P}(\Theta, X \mid Y) = \mathcal{P}(Y \mid \Theta, X)\mathcal{P}(X \mid \Theta)\mathcal{P}(\Theta) \tag{3.11}$$

The MCMC method creates a Markov chain over couple (Θ, X): given the initial draws $X^{(0)}$ and $\Theta^{(0)}$, the g-th draws are produced through iteration as:

$$X^{(g)} \sim p\left(X \mid \Theta^{(g-1)}, Y\right)$$
$$\Theta^{(g)} \sim p\left(\Theta \mid X^{(g)}, Y\right)$$

The sequence of random variables $\left\{\Theta^{(g)}, X^{(g)}\right\}_{g=1}^{G}$ obtained is a Markov chain, whose distribution converges to $p(\Theta, X \mid Y)$. The MCMC method is implemented through the use of various algorithms like the *Metropolis-Hastings algorithm*. A candidate draw is drawn from a chosen probability density and accepted or rejected using a certain criterion. The criteria is selected in order to produce random samples shaping a Markov chain with the proper equilibrium distribution.
Given the sample $\left\{\Theta^{(g)}, X^{(g)}\right\}_{g=1}^{G}$ from the joint posterior, parameter and state variable estimation can be performed with Monte Carlo method. If $f(\Theta, X)$ is a function satisfying technical regularity conditions, the Monte Carlo estimates

$$E[f(\Theta, X) \mid Y] = \int f(\Theta, X)p(\Theta, X \mid Y)dXd\Theta \approx \frac{1}{G}\sum_{g=1}^{G} f\left(\Theta^{(g)}, X^{(g)}\right)$$

We can, in addition, analyse two types of convergence for $G \rightarrow \infty$:

- The convergence of the distribution of the Markov chain to $p(\Theta, X \mid Y)$.
- The convergence of the partial sums

$$\frac{1}{G}\sum_{g=1}^{G} f\left(\Theta^{(g)}, X^{(g)}\right)$$

[6] Markov chain Monte Carlo
[7] Stochastic volatility

to the conditional expectation $E[f(\Theta, X) \mid Y]$. Both types of convergence are guaranteed by the Ergodic Theorem for Markov Chains, since MCMC algorithm verifies the statement's holding conditions.

The MCMC algorithms used are explained in details in Appendix A.

SVCJ Calibration

The equity price, S_t, and its stochastic variance, V_t, jointly solve

$$dS_t = S_t\left(r_t + \eta_v V_t\right)dt + S_t\sqrt{V_t}dW_t^s(P) + d\left(\sum_{j=1}^{N_t(P)} S_{\tau_j-}\left(e^{Z_j^s(P)} - 1\right)\right)$$

$$dV_t = \kappa_v\left(\bar{v} - V_t\right)dt + \sigma\sqrt{V_t}dW_t^v(P) + d\left(\sum_{j=1}^{N_t(P)} Z_j^v(P)\right)$$

where $W_t^s(P)$ and $W_t^v(P)$ are correlated Brownian motions, $\operatorname{corr}\left(W_t^s(P), W_t^v(P)\right) = \rho$, $N_t(P) \sim \text{Poisson}(\lambda), \tau_j$ are the jump times, $Z_j^s(P) \mid Z_j^v \sim \mathcal{N}\left(\mu_s + \rho_s Z_j^v, \sigma_s^2\right)$ are the return jumps, $Z_j^v(P) \sim \exp\left(\mu_v\right)$ are the volatility jumps, and r_t is the spot interest rate.

The MCMC estimation scheme uses the Euler discretization:

$$Y_t = \mu + \sqrt{V_{t-1}}\varepsilon_t^y + Z_t^y J_t$$
$$V_t = \alpha + \beta V_{t-1} + \sigma_V\sqrt{V_{t-1}}\varepsilon_t^v + Z_t^v J_t$$

- Where $Y_{t+1} = \log\left(S_{t+1}/S_t\right)$ is the log return, $\alpha = \kappa\bar{v}, \beta = 1-\kappa$ and $\varepsilon_t^y, \varepsilon_t^v$ are $\mathcal{N}(0,1)$ with ρ correlation.
- J_t is the jump Bernoulli variable that represents the occurrence of jumps generated by the poisson process such as $\mathbb{P}(J_t = 1) = \lambda$ in other terms $\mathbb{P}(J_s = 1; s \in [t, t+dt]) = \mathbb{P}(N_{t+dt} - N_t = 1) = \lambda dt$

We define the parameter vector:

$$\Theta = \{\mu, \mu_y, \sigma_y, \lambda, \alpha, \beta, \sigma, \rho, \rho_j, \mu_v\}$$

$X_t = \{V_t, Z_t^y, Z_t^v, J_t\}$ being the chain at time t, composed of the latent variance, jump sizes and jump occurrence respectively.

Following (Hou, Wang, Chen & Härdle, 2020), the prior of parameters are chosen as follows:

$$\begin{array}{cc} \lambda \sim Be(2,40) & \sigma^2 \sim \mathcal{IG}(2.5, 0.1) \\ \mu \sim \mathcal{N}(0,25) & \mu_y \sim \mathcal{N}(0,100) \\ \sigma_y^2 \sim \mathcal{IG}(10,40) & \rho \sim U(1,1) \\ \rho_j \sim \mathcal{N}(0,0.5) & \mu_v \sim \mathcal{IG}(10,20) \\ (\alpha,\beta) \sim \mathcal{N}\left(0_{2\times1}, I_{2\times2}\right) & \end{array}$$

Parameters	Mean	St. deviation
μ	0.2162	0.3293
μ_y	0.0521	0.2346
σ_y	2.4068	0.0974
λ	0.1126	0.1933
α	0.0188	0.1946
β	-0.2524	0.1159
ρ	0.5432	0.05791
σ	0.0439	0.0790
ρ_j	-0.6430	0.1214
μ_j	0.9429	0.2130

Table 3.1: Estimated parameters of the SVCJ model using MCMC

Be and $\mathcal{IG}$ are the Beta Distribution and the Inverse Gaussian.
The posterior mean estimating Θ is really robust to different changes in variance of the prior distributions. One must say that the posterior for all parameters except σ and ρ are conjugate[8]. Therefore, we can see that posterior for the jump sizes posterior law are a normal distribution for Z_t^y and a truncated normal distribution for Z_t^v. The posterior law for J_t is a Bernoulli distribution. However for ρ, σ^2 and V_t the posteriors are nonstandard distributions. That is why sampling for these parameters should be done using the Metropolis-Hastings algorithm. We use independence sampling for σ^2 and random-walk method for ρ and V_t. The algorithms need 5000 iterations in order to estimate posteriors distributions (In our case we need only the first two moments). To decrease the impact of the starting value we neglect the first 1000 iterations. These non considered iterations are called the burn-in.

The Stochastic Volatility model with Correlated Jumps can reproduce unexpected jumps occurring due to periods of high volatility. This periods of high volatility and jumps are usually caused by events happening in the financial market or news destabilizing the cryptocurrency one. A jump is considered to be occurred at t if the estimated jump probability is sufficiently large exceeding a certain threshold ζ,

$$\hat{J}_t > \zeta, \quad t = 1, 2, \ldots, T$$

where $\hat{J}_t \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^{N} J_t^i$ is an estimation indicating the posterior probability of the occurrence of a jump at time t. ζ is chosen empirically so that the number of inferred jump times divided by the number of observations is approximately equal to the estimate of λ.

3.5.3 Calibration results

The calibration process yielded the parameters in table 3.1: The estimate of μ is positive.

[8]Being conjugate is having the same density function yet with different parameters

The correlation between returns and volatility ρ is significant and positive. The positive correlation can also be related to the noise trading feature explained in the first chapter. Because the BTC market is highly unregulated but also due to the fact that the BTC price is driven by emotion and sentiment, the speculative behaviour can be explained by the "noise trader" theory. The positive relation might result from the fact that BTC investors irrationally act on noise as if it were information that would give them an edge. We can see also that the mean of the jump size of the volatility μ_v is significant and positive. The jump intensity is also significant. In summary, the SVCJ model fits the data well and the significance of the jump parameters relatively explains the need of this model.

3.5.4 Pricing cryptocurrency options

The pricing of the cryptocurrency vanilla options is done using Crude Monte Carlo simulations[9] using:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}C(T) \mid \mathcal{F}_t\right]$$

As expected the SVCJ model outperformed the previous models with an RMSE of 8.12 %. We were able to reproduce the SVCJ volatility surface in fig 3.8

We can see that this model represents well the volatility surface plus we can observe thanks to the error surface that we relatively solved the short-term skew fitting problem of the Heston model.

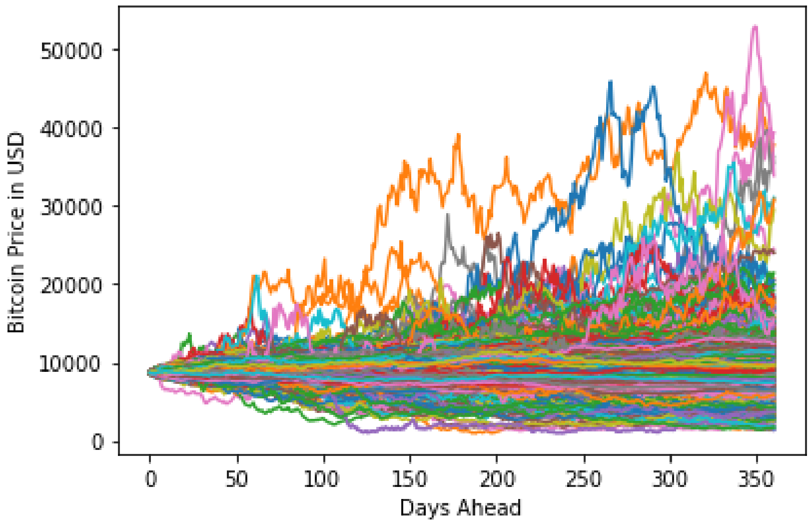
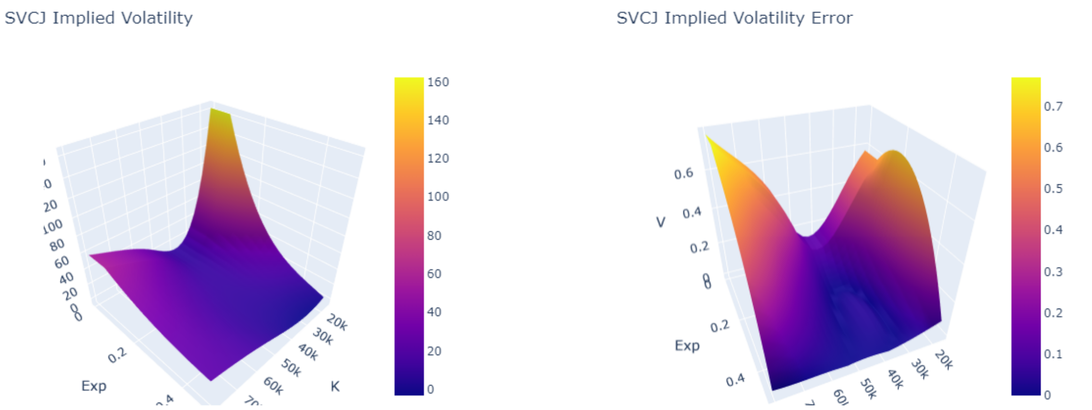


Figure 3.7: Monte Carlo simulations of the price process

Martingale hypothesis testing

Since the pricing of options is done using Monte Carlo, under the risk neutral probability the process should be a martingale. To ensure that the pricing is done correctly, a martin-

[9]see appendix B



(a) SVCJ Implied volatility surface (b) SVCJ Implied volatility surface error

Figure 3.8: SVCJ Implied volatility surface (on 30/07/2021)

gale test could be executed to verify the latter hypothesis.

The test Let a time series (y_t) be given, and let $(\mathcal{F}_t)$ be a filtration to which (y_t) is adapted. The null hypothesis of interest is that (y_t) is a martingale process with respect to the filtration $(\mathcal{F}_t)$, i.e.

$$H_0 : \mathbf{P}\left(\mathbf{E}\left(y_t \mid \mathcal{F}_{t-1}\right) = y_{t-1}\right) = 1$$

for each $t \geq 1$, where $\mathbf{E}\left(\cdot \mid \mathcal{F}_{t-1}\right)$ denotes as usual the conditional expectation given $\mathcal{F}_{t-1}$. Since the process is Markovian $\mathbf{E}\left(y_t \mid \mathcal{F}_{t-1}\right) = \mathbf{E}\left(y_t \mid y_{t-1}\right)$. Using,

$$\mathbf{E}\left(\triangle y_t \mid y_{t-1}\right) = 0 \text{ a.s. iff } \mathbf{E}\triangle y_t 1\left\{y_{t-1} \leq x\right\} = 0 \text{ for almost all } x \in \mathbf{R}$$

the basis of the test statistics for the martingale hypothesis,

$$Q_n(x) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \Delta y_t 1\left\{y_{t-1} \leq x\right\}.$$

The test statistic, therefore, considered is Kolmogorov-Smirnov one:

$$S_n = \sup_{x \in \mathbf{R}} \left|Q_n(x)\right|.$$

Asymptotic Critical Values of S_n(*Park &Whang*,2004)

sig. level (α)	0.99	0.95	0.90	0.10	0.05	0.01
S_n	0.612	0.765	0.865	2.119	2.388	2.911

The value of the statistic computed is 0.724 wich indicates the martingality of the process with 95% confidence level.

3.5.5 The impact of jumps in volatility and returns

Some of the impacts are similar to other standard markets. See (Eraker, Johannes & Polson, 2003). As we stated in the previous section, stochastic volatility models are efficient when it comes to long maturities as they can reproduce the implied volatility in a realistic way without having to excessively re-calibrate parameters. However this is not true for short maturities As seen in the Merton model, this problem is not so relevant. Therefore, adding jumps in returns to stochastic volatility models allows to calibrate the implied volatility surface across different strikes and maturities using parameters without an explicit time dependence.

The increments of the price process being not totally independent results actually in sufficiently time-stable calibration parameters; The parameters being time-stable is actually very useful since this way forward smiles can be estimated without being influenced by present smile.

Adding Jumps to the price process (return process) helps explain empirically explain a relevant portion of the total variance since they represent rare events of large sizes. The jump sizes are actually temporary in contrast to volatility jumps that lean to persistent and rapid shocks in the conditional volatility of returns as they are driven by a Brownian motion. Computing these jumps contributions can help us hedging jump risk by estimating appropriate premiums in the time of stressed markets.

As discussed before, jumps of volatility are instantaneous on the volatility process itself yet persistent on the returns process that is due to the property of mean-reversion of the volatility process that make it mean-revert back to its long-run level. Yet the effect on the price process is usually long lasting. In periods of market stress the contribution of both jump processes cannot be ignored as they are important in the determination of risk premiums. Those jumps are usually incisive than diffusion processes to produce large movements observed in financial crashes.

Finally these jumps help shape the volatility surface as they make its slope curve steeper. Volatility jumps help incline volatility for ITM options and make the slope even steeper.

Conclusion

The SVCJ model performed well compared to the previous ones. All the parameters that we calibrated turned out to be statistically significant which justifies the use of both volatility and price jumps and the stochastic nature of the volatility process. A fractal analysis is conducted in the next chapter to see whether we ought to use rough volatility model to better understand the bitcoin behaviour.

Chapter 4

Bitcoin market's modelling: A fractal market

"A fractal is a way of seeing infinity."

Benoit Mandelbrot

4.1 Introduction

Rough models aren't rough.

The variation of a Brownian motion is infinite. This means that a Brownian motion can oscillate so fast in a short period of time that his variations in a finite time interval tends to infinity. A Brownian motion's derivative over time is infinite in every single point. Meaning that if we know the path we would never know it's momentum[1] yet if we know it's law of transition which is analogous to its momentum we can not know it's position: Heisenberg's uncertainty principle hits again.

I was actually dazzled by a saying of my stochastic calculus professor "The essence of Ito calculus is that space is the square root of time". Studying financial markets on a macroscopic or microscopic scale is all about the trading time. Time is not an absolute truth. There isn't a supreme clock ticking for the whole universe. Time actually depends on the reference and on the scale of events. Bachelier was actually a maverick when describing the price process as a Brownian motion. As he implied by that that financial market are random and therefore cannot be modeled as a deterministic function of a constant time. Actually there is a lot happening in the financial market as every single agent is affecting the prices on an atomic scale that actually for the price process point of view the clock ticks slower and a lot can happen in a minor time interval.

[1] if we fix ω the brownian motion's "speed" is simlply its differential with respect to time which is infinite. If we don't know it's position means ω is variable we know the law of it's increments yet we don't know it's trajectory

It wasn't until the black monday that we realised Bachelier and his colleges missed something too. The trading time is not always constant sometimes the market is calm: nothing is happening. Other times, in a case of market turbulence everything can be chaotic and everything is possible including a market crash. In this case, even the slower time rate of the Brownian motion is not sufficient to reproduce the markets turbulent state. Therefore another process or another more general definition of time is needed: The multi-fractal time. To better explain this idea, I must cite the following passage of Benoit Mandelbrot's book The Misbehavior of Markets: A Fractal View of Financial Turbulence

> "*On occasion, trading is fast. Scores of news ,items are flitting across the electronic "crawl" on the bottom of the screen. Colleagues are waving and shouting all around. Phones are ringing. Customers are zapping electronic orders. The volume of trades is climbing, and prices are flying by. On such days are fortunes won or lost. Time flies. Then there are the slow times. No news, only tired reports from the in-house financial analysts to chew over. The customers seem to be on holiday. Trading is thin. Prices are quiet. No big money to be made here; might as well go for a long lunch. Time hangs heavy.*
> "

The fractional Brownian motion is not just a rougher stochastic processes yet it's a more general approach that makes them fit reality in every single scale and time. The rough models aren't just rough, they are rich enough to incorporate every state of the market.

This chapter was motivated by the work of Takaishi and Celeste, Corbet and Gurdgiev

4.2 Cryptocurrency and the fractal market hypothesis (FMH)

The Efficient market hypothesis (EMH) was developed through the pioneering work of Osborne (1959) who empirically showed that stock prices follow a random walk. EMH implies that the price process reflects all available information and therefore it is impossible for markets participants to consistently outperform the market on risk adjusted basis. Although a lot of today's models are based on EMH, empirically observed properties of the financial markets, show that this theory is far from being valid.

In his book, Peters outlined a new theory based on empirical studies of the different markets: The fractal market hypothesis (FMH). This theory states that the market contains many investors with different investment horizons. Therefore the trading activity of them is based on different information sets. For instance, investors with longer-term horizons base their decisions on fundamental information, yet shorter-term ones rely on technical aspects. This fractal structure helps maintaining the stability of the market since each investment horizon provides liquidity to the other and thus fair prices are traded. During turbulent times, long-term investors begin lose confidence in their information, their investment horizon shortens, and thus the overall investment horizon of the market loses its structure. The market, thus, becomes unstable because trading is based on the same information set, interpreted in the same way. Trading is therefore based on exogenous information like the news : Good news, for example, raises confidence in a certain stock or index which drives the price upwards. This uniform nature of investment horizons causes liquidity to dry up and therefore volatility to increase, because most of the trading is on one side. Volatility being important, more short term investors are driven to the market which causes this time horizon to be even more dominant. Hence the self exciting character described in Zipf's work. Ultimately, the long term gains certainty and the market returns to stability as investment horizons widen and diversify. In this section, we will try to prove the existence of this fractal behaviour in cryptocurrency markets, we will use the Bitcoin price evolution[2] as long as the 5 last years daily historical volatility[3], for our empirical study.

4.2.1 Evidence of cryptocurrencies fractal behaviour

During times of low uncertainty, markets will follow the assumptions of the EMH with Gaussian distributions and finite variance statistics. When uncertainty increases, markets exhibit fat-tailed risks and unstable variance. In order to capture and prove these fractal dynamics we investigated several aspects of the multifractal analysis.
Among the most relevant properties of Mandelbrot's fractal geometry, the self-similarity and fractal dimension: In presence of fractal dynamics, each scale looks similar but not identical to the others, making data scale invariant or self-similar.
In the usual Euclidean geometry, objects' dimension, namely how an object fill its space are integers whereas the dimension of a fractal object is fractal. Fractal objects are actually infinite in a finite portion of the higher dimension so they actually fill even more than the space they are in but not enough to actually be an object in a higher dimension. That's why we consider a brownian motion to have an infinite variation yet it is still a one dimensional curve not a two dimensional object.

Clearly to measure an object dimension we need to have a more general measure. This more general definition is the Hausdorff-Besicovitch dimension. In order to simplify the work done to analyse the bitcoin's price time series, we will focus on the Hurst exponent knowing that there is a straight equation that relates both quantities:

$$D = 2 - H$$

Where D is the fractal dimension and H is the Hurst exponent. When D is close to 1 the market shows persistence and a long-memory behaviour however when D is close to 2 the market is anti-persistant and show a mean-reversion character which is translated by short trading horizons and a turbulant behaviour.

[2]Source: Bloomberg
[3]Source: Bitstamp.

The general Hurst exponent and fractal dimension

The MF-DFA analysis For the Multifractal analysis one should follow the coming steps:

1. Determine the quantity $Y(i)$,

$$Y(i) = \sum_{j=1}^{i}(r(j) - \langle r \rangle)$$

where $\langle r \rangle$ is mean of returns.

2. Divide $Y(i)$ into N_s equal and segments that are not overlapping s, where $N_s \equiv \text{int}(N/s)$. Knowing that time series' length is not generally a multiple of s we use the same procedure starting from the end of Y to use the residual. In total, we have $2N_s$ segments.

3. Compute the variance

$$F^2(\nu, s) = \frac{1}{s}\sum_{i=1}^{s}(Y[(\nu-1)s+i] - P_\nu(i))^2$$

for each segment $\nu, \nu = 1, \ldots, N_s$ and

$$F^2(\nu, s) = \frac{1}{s}\sum_{i=1}^{s}(Y[N-(\nu-N_s)s+i] - P_\nu(i))^2$$

for each segment $\nu, \nu = N_s+1, \ldots, 2N_s$. we substruct $P_\nu(i)$, which is a fitting polynomial (order 2), to get rid of the local trend.

4. Average over all segments and obtain the q th order fluctuation function

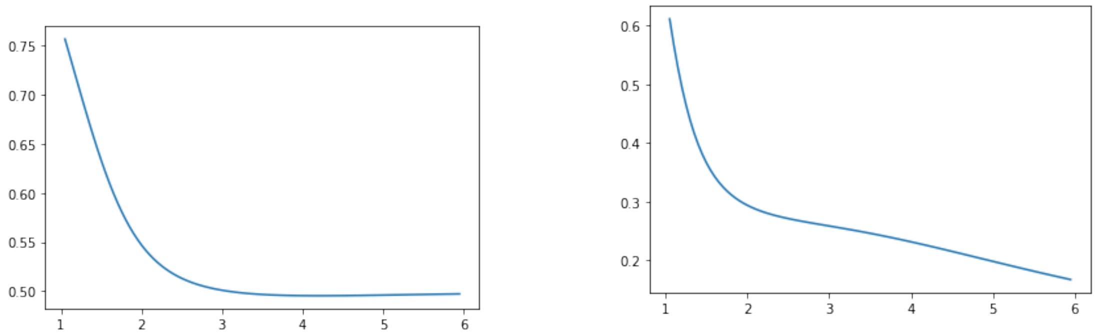
$$F_q(s) = \left\{\frac{1}{2N_s}\sum_{\nu=1}^{2N_s}\left(F^2(\nu, s)\right)^{q/2}\right\}^{1/q}$$

For $q = 0$, the averaging procedure in the previous equation cannot be directly applied. Instead, we employ the following logarithmic averaging procedure.

$$F_0(s) = \exp\left[\frac{1}{4N_s}\sum_{\nu=1}^{2N_s}\ln\left(F^2(\nu, s)\right)\right]$$

5. If the time series $r(i)$ are long-range power law correlated, $F_q(s)$ is expected to be the following functional form for large s.

$$F_q(s) \sim s^{h(q)}$$



(a) Daily price changes from 2013 to 2021

(b) 1-min price changes from 20/09/21 to 30/09/21

Figure 4.1: The general Hurst exponent h(q)

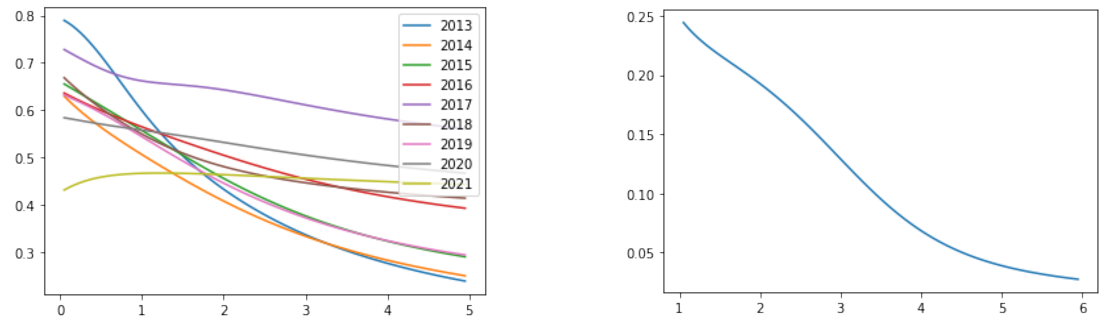
The scaling exponent $h(q)$ is the generalized Hurst exponent. For $q = 2$, $h(2)$ corresponds to the well-known Hurst exponent H If $h(2) < \frac{1}{2}$, the time series is anti-persist and if $h(2) > \frac{1}{2}$, it is persistent. For $h(2) = \frac{1}{2}$, the time series becomes a random walk.
After plotting the slope of $log(F_q(s)) \sim h(q)\, log(s)$ in 4.1 we can see that the slope changes depending on q, which indicates the multi-fractal property of the price process.

$h(2)$, being the standard Husrt exponent, can show that for the daily data set going from the year of 2013 to 2021 the exponent is equal to 0.56. Following (Weron, 2002), the functional of the confidence interval is

$$0.5 \pm e^{(-2.93 \log(M)+4.45)}$$

Where $M = \log_2(N)$. N being the sample size. The confidence interval is so narrow that null hypothesis of the process following a Brownian motion is rejected.
This means that curve of Bitcoin price shows persistence in general. Taking a look on the 1-min data, we notice a coefficient of 0.3 that shows a certain mean-reversion and a turbulent state of the market. When investigating the price process during this period of time we can see a mean reversion character and a slight increase in the historical volatility. When applying the same analysis to the yearly data. We notice that for every year a multi-fractal behaviour is captured. We can also see that some years exhibit anti-persistence. For example, the last year, the returns process shows roughness which is consistent with the large prices changes that bitcoin knew and its sensitivity to external events and market news which implies the dominance of short-term trading horizons. A variable Hurst parameter, fig 4.3, also suggests that the dynamics of bitcoins price vary overtime and a more general model should be used to capture this variable behaviour.

Applying the Multi-fractal analysis to the historical volatility, we see that it also exhibits a multifractal behaviour. The Hurst exponent is less than $\frac{1}{2}$ which is consistent with results empirically observed for other assets. This results support the use of a rough volatility model.



(a) Daily price changes exponent by year

(b) Volatility

Figure 4.2: The general Hurst exponent h(q)

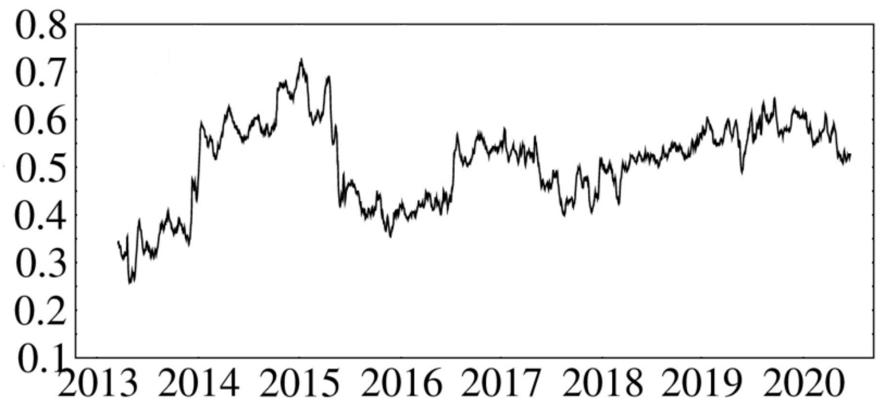


Figure 4.3: The rolling Hurst exponent (Takaishi, 2021)

Wavelet analysis

The strength of the wavelet analysis is its ability to visualize the underlying's information both in the frequency and time domains, thus providing information about the price evolution across different frequency and time scales. In this part we will reproduce the analysis done by (Kristoufek, 2013) for standard markets on the Bitcoin's one.

Wavelet: A wavelet $\psi_{u,s}(t)$ is an integrable function defined as

$$\psi_{u,s}(t) = \frac{\psi\left(\frac{t-u}{s}\right)}{\sqrt{s}}$$

s being the scale, u the location and t the time. Any time series can be reconstructed back from its wavelet transform if the admissibility condition

$$C_\Psi = \int_0^{+\infty} \frac{|\Psi(f)|^2}{f} df < +\infty$$

holds, where $\Psi(f)$ is the Fourier transform of a wavelet.
Wavelet satisfy $\int_{-\infty}^{+\infty} \psi(t)dt = 0$ and $\int_{-\infty}^{+\infty} \psi^2(t)dt = 1$. To obtain the continuous wavelet transform $W_x(u,s)$, a wavelet $\psi(.)$ is projected onto the examined series $x(t)$ so that

$$W_x(u,s) = \int_{-\infty}^{+\infty} \frac{x(t)\psi^*\left(\frac{t-u}{s}\right)dt}{\sqrt{s}}$$

where $\psi^*(.)$ is a complex conjugate of $\psi(.)$. Importantly, the continuous wavelet transform decomposes the series into frequencies and can then reconstruct the original series so that there is no information loss, and energy of the examined series is maintained as well, i.e.

$$x(t) = \frac{\int_0^{+\infty}\int_{-\infty}^{+\infty} W_x(u,s)\psi_{u,s}(t)duds}{s^2 C_\Psi}$$
$$\|x\|^2 = \frac{\int_0^{+\infty}\int_{-\infty}^{+\infty} |W_x(u,s)|^2 duds}{s^2 C_\Psi}$$

where $|W_x(u,s)|^2$ is the wavelet power at scale $s > 0$. The wavelet used in our analysis is the Morelet wavelet, it has been chosen in accordance with the existing literature and is defined as follows with the existing literature

$$\psi(t) = \frac{e^{i\omega_0 t - t^2/2}}{\pi^{1/4}}$$

with $\omega_0 = 6$, to provide a good balance between the time and frequency localization.

According to the FMH arguments, we should observe increased power at low scales or high frequency during the critical periods. Moreover, we might observe a changing structure of variance across frequencies before the turbulences due to the changing structure of investors' activity and thus wavelet analysis might be a good tool to adapt the trading activity to the market changes.

Regions of highest power are those characterised by high volatility. Important and significant power is detected at high frequencies confirming, thus, the dominance of high frequency trading. We can also see some power detected for lower frequencies implying that in those periods, from 2013 to mid 2014, investors with multiple time horizons contributed to the volatility generated. However, in the next periods we can notice lack of power in the part corresponding to mid 2014 to 2017. The wavelet power spectrum of Bitcoin daily returns at the end of 2017 shows a clear dominance at high frequency during the moments of high volatility. Finally we can see a shadow of this behaviour in the last year which explains the turbulence of this market and goes along with the multi-fractal analysis. The turbulent times are characterised by the dominance of short horizons because the efficient market clearing of orders is not possible due to the lack of liquidity.

These results confirm the fractal market hypothesis and therefore a fractal or multifractal model should be used in order to accurately model the dynamics of Bitcoin and other cryptocurrency markets.

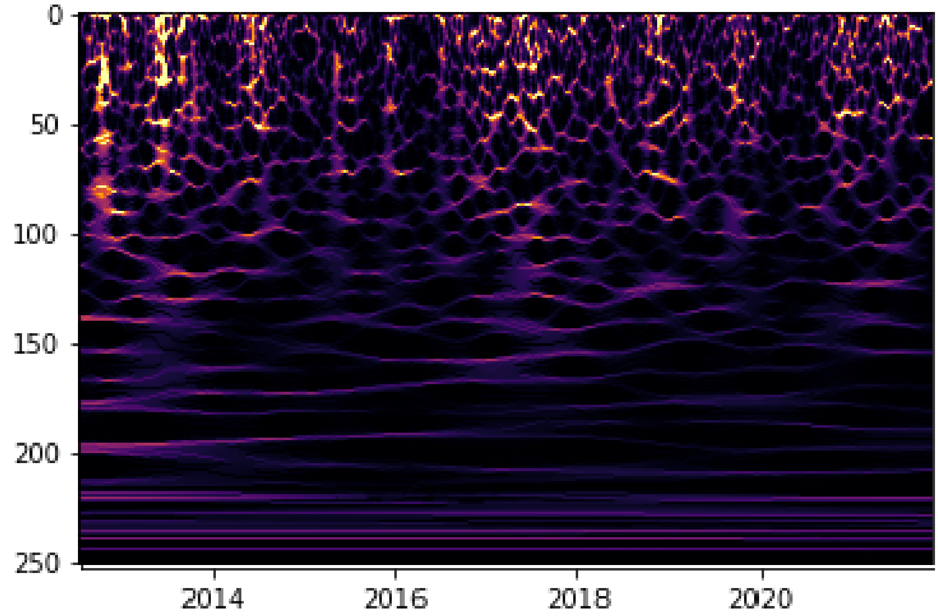


Figure 4.4: CWT spectogram of bitcoin daily returns (Y axis represent the trading period)

Conclusion

Both the Muti-detrended fractal analysis and the wavelet analysis show that the Bitcoin price dynamics present fractal behaviour, not only that but the Multi-detrended fractal analysis show a multi-fractal behaviour. Therefore, it is interesting to try a rough stochastic model as it my capture the rough behaviour of cryptocurrency.

4.3 Rough volatility model

4.3.1 Fractional Brownian motion

The fractional Brownian motion is a generalisation of the Brownian motion as it is a fBM when H=$\frac{1}{2}$

Definition 4 *The fractional Brownian motion (* fBm *) with Hurst parameter* H *is the only process* W^H *to satisfy :*

- *Self-similarity :* $\left(W_{at}^H\right) \overset{\mathcal{L}}{=} a^H \left(W_t^H\right)$
- *Stationary increments :* $\left(W_{t+h}^H - W_t^H\right) \overset{\mathcal{L}}{=} \left(W_h^H\right)$
- *Gaussian process with* $\mathbb{E}\left[W_1^H\right] = 0$ *and* $\mathbb{E}\left[\left(W_1^H\right)^2\right] = 1$

Proposition 1 *The absolute moments satisfy*

$$\mathbb{E}\left[\left|W_{t+h}^H - W_t^H\right|^q\right] = K_q h^{Hq}$$

Proposition 2 *For all* $\varepsilon > 0, W^H$ *is* $(H-\varepsilon)$*-Hölder a.s.*

Proposition 3 *The Mandelbrot-van Ness representation:*

$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}}\right) dW_s$$

One can neglect the second term as it barely influences the formula

Assumptions

4.3.2 Rough Heston model

The model we will use is going to be a rough Heston model. The calibration of the Hurst coefficient will adapt the model to the varying nature of volatility dynamics.
We consider the same filtered probability space as before. Let S_t be the the price process and V_t be the volatility process. The dynamics of the rough Heston model are described as follows.

$$\begin{gathered} d\log S_t = \mu dt + \sqrt{V_t} dW_t^{(s)} \\ V_t = \xi_0(t) + \frac{\nu}{\Gamma\left(H+\frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}} \sqrt{V_s} dW_s^{(V)} \\ d\left\langle W_t^{(S)}, W_t^{(V)} \right\rangle = \rho dt \end{gathered} \tag{4.1}$$

where $\nu > 0, H \in \left(0, \frac{1}{2}\right)$. The rough character is due to the the kernel $(t-s)^{H-\frac{1}{2}}$. Let be $\alpha = H + \frac{1}{2}$.

And where $\xi_0(t) = E[V_t], t \geq 0$, which means the initial forward variance curve is an input[4]. Same as the Heston model the rough Heston model has an affine structure. So, solving a fractional Riccati equation, we can get the characteristic function:

$$E\left[e^{ia\log(S_T)}\right] = \exp\left(ia\log(S_0) + \int_0^T D^\alpha h(a, T-u)\cdot\xi_0(u)du\right)$$

D^α the Riemann-Liouville fractional derivative operator of order α,i refers to the imaginary number, and $h(a,\cdot)$ solves the following fractional Riccati equation:

$$D^\alpha h(a,t) = -\frac{1}{2}a(a+i) + i\rho\nu a h(a,t) + \frac{1}{2}\nu^2 h^2(a,t), I^{1-\alpha}h(a,0) = 0, \tag{4.2}$$

$I^{1-\alpha}$ being the Riemann-Liouville fractional integral operator of order $1-\alpha$.To refresh one's memory: for a function f:

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}f(s)ds, \quad I^{1-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f(s)ds.$$

Data simulation

When the correlation factor is negative we use the fast rational approximation done by (Gatheral & Radoičić, 2019) and (Rømer, 2020) of the equation solution. This gives us the characteristic function and then Heston pricing formulas are used. Other regions will be priced using Monte Carlo method.

To simulate the fractional Brownian motion(fBM) we can use two methods. The *Cholesky* method with time comlexity of g and the rDonsker method with time complexity $\mathcal{O}(N\,log(N))$. The simulation is based on the standard random motion simulation. Meaning we will be simulating a gaussian vector representing the increments of the process. However, in the fBM they are not independent. To explain the simulation process we start with the Cholesky method. Since we know the formula to compute the moments between different increments. We can compute Γ, the variance-covariance matrix.

$$\begin{aligned}\Gamma(t,s) &= \operatorname{Cov}(B_H(t), B_H(s)) = \frac{C^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)\\ \gamma(t-s) &= \operatorname{Cov}(B_H(t+1) - B_H(t), B_H(s+1) - B_H(s))\\ &= \frac{C^2}{2}\left(|t-s-1|^{2H} - 2|t-s|^{2H} + |t-s+1|^{2H}\right).\end{aligned}$$

When discretized at times i/N, for $i = 0,\ldots,N-1$.

$$(\Gamma)_{i,j} = \Gamma\left(\frac{i}{N}, \frac{j}{N}\right), \quad \text{for } i,j = 0,\ldots,N-1.$$

[4]This curve can be obtained from quotes of variance swaps which in turn can be obtained from a full surface of European call and put options

Define Γ' as the matrix Γ deprived of its first row and its first column. Since Γ' is a symmetric definite positive matrix, it admits a Choleski decomposition $\Gamma' = LL^t$, where L is a lower triangular matrix. Thus simulating a sample of a fBm at times i/N for $i = 1, \dots, N-1$ is equivalent to generate a vector Z of $(N-1)$ standard independent Gaussian variables and apply the product LZ. Indeed, LZ is a centered Gaussian vector and $\mathbb{E}\left((LZ)(LZ)^t\right) = \Gamma'$. Define $\widetilde{B} = \left(0, (LZ)^t\right)^t, \widetilde{B}$ is a sample of a fBm at times i/N for $i = 0, \dots, N-1$. This method is the only one exact in theory, but due to a computational complexity of order $\mathcal{O}\left(N^2\right)$ at best we use the rDonsker method.
To calibrate the model a Neural network approach would be used. 50000 call and put samples, of the rough Heston model has been simulated for the training of the Network. The different parameters where sampled using the following laws as used in (Rømer, 2020)

	Rough Heston		
Par.	Distr.	a	b
H	$\mathcal{U}$	0	0.5
ν	$\mathcal{U}$	0.10	1.25
ρ	$\mathcal{U}$	-1	1
$\sqrt{\xi}$	G_ξ	0.05	1

Calibration of the model

To calibrate the model we build a neural network representation of it

Neural networks Since neural networks are very efficient when approximating multivariate function $F : \mathbb{R}^N \to \mathbb{R}^M$. In order to approximate the pricing formula in a way that is fast to compute, thanks to the chain rule in computing derivatives, we will consider a fully connected feed forward neural network. See the appendix C for further details on neural networks.

The neural network designed for approximating implied volatilities on expiry-slice in 5 equal intervals and that for a model with n input parameters. Parameters are denoted θ_i for $i = 1, \dots, n$ and implied volatilities σ_i. As hyperparameters for our network we used: 3 hidden layers, the Adam algorithm as an optimizer, Elu as our activation function and root-meansquared-error as the loss function. We also scale the inputs and outputs. We fixed the number of neurons to 200 neurons in each layer. For the actual training, we let the optimizer run for 700 epochs using a batch size of 2000.
Training our model on our simulated dataset got us the following training accuracy:

Model/percentile	**50%**	**75%**	**95%**	**99%**	**99.9%**	**99.99%**	Max
Rough Heston	0.14%	0.24%	0.81%	1.94%	6.21%	14.07%	84.23%

The testing accuracy is a little bit higher but didn't exceed 2.1% for the 75% percentile error.

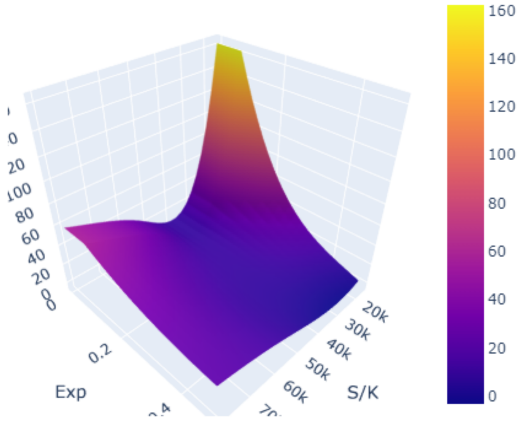


Figure 4.5: Implied volatility surface (on 30/07/2021)

Calibration For the integer pair (i, j) referring to the j^{th} quoted strike of the i^{th} expiry. σ_{ij}^{mid} denotes the implied volatility. Let $\sigma_{ij}^{\text{model}}(\theta)$ be the implied volatility under rough Heston model with parameter vector θ. The optimisation problem is then the minimisation of the following RMSE with respect to the parameters vector.

$$\text{RMSE}(\theta) := \sqrt{\sum_{(i,j)} w_{ij} \left(\sigma_{ij}^{\text{model}} - \sigma_{ij}^{\text{mid}}(\theta)\right)^2}$$

where the sum is over all contracts (i, j) from the particular calibration date and where w_{ij} denote some specific weights that are normalised so $\sum_{(i,j)} w_{ij} = 1$. The weights represent the number of contracts for a certain contract specification and liquidity.
The parameters of the model are then obtained using the **back propagation** aspect of the deep learning using the trained neural network.

Results The back propagation process used on our set for the end of July 2021 yielded the following parameters:

Rough Heston cal. params

	H	ρ	ν
Params	0.21	0.72	1.68

We can see that we have the same aspects as the other models. The rough Heston model captures the positive skew. The hurst exponent $H < \frac{1}{2}$ which indicates the mean-reversion and the rough character of the volatility.

Using these parameters trying to reproduce the volatility surface yielded the result in fig 4.5

We can see that this model better replicates the volatility surface than the previous ones. The RMSE for this model was 3.78%

Conclusion

The reason the rough volatility model surpassed the other models is that it captures the behaviour of market using far less parameters. Capturing the turbulence of the market groups the behaviour of several parameters in the stochastic volatility model with co-jumps. Finding efficient ways to calibrate those models can revolutionize the quantitative finance field, as we can reproduce the market dynamics with a more general point of view, fewer parameters and less monitoring.

4.3.3 Multi-fractal Model

Although the last model outperformed the non fractional ones, it still can benefit from some improvements. As we have seen in the first part the price and volatility process exhibits multifractality. After examination of the Hurst exponent in 4.3 we might see that the Hurst process follows a certain process and varies across the time-frame. It is, therefore to use Mandelbrot's generalization of the trading time and substitute the fBM with the multi-fractional Brownian motion (mBm).
the mBm is a Gaussian process for which the pointwise Hölder exponent may be tuned at each point: multifractional Brownian motion (mBm) is such an extension, obtained by substituting the constant parameter $H \in (0, 1)$ with a regularity function H ranging in $(0, 1)$:

$$W_t^{H(t)} = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H(t)}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H(t)}} - \frac{1}{(-s)^{\frac{1}{2}-H(t)}} \right) dW_s$$

The Hölder exponents of mBm are prescribed almost surely: almost surely, the pointwise (resp. local) Hölder exponent at t is the minimum between H(t) and the pointwise (resp. local) exponent of H at t. Multifractional Brownian motion is our prime example of a stochastic process with prescribed local regularity. Other approaches exist in the literature like the fractional Brownian motion with a regime switching Hurst exponent in (Liu, Di Matteo & Lux, 2007) where the Hurst exponent is a continuous Markov chain. However, pricing under this model requires really complex computations.

Finally, and most importantly, investigating the rolling Hurst exponent in 4.3 we can notice a sort of seasonality that might help us predict the behaviour of the market and thus adapt our risk management and trading strategy. As we can see that a certain increase of the coefficient happens every 3 to 4 years, although that this effect is decreasing over the time. This is might be due to the halving principle that affects the equilibrium between supply and demand. The halving have the same effect as decreasing the supply. It makes the generation of bitcoins harder ad therefore makes it rarer which drives the price up.

Driving the price up increases the volatility and make the system enters a turbulent state. The Hurst exponent stabilises gradually after that.

The cryptocurrency market is widely driven by sentiment, beside catching this behaviour using an advanced multi-fractal model one can also think of adding a stochastic quantity representing the sentiment of the market and calibrating its weight that when market is turbulent (Hurst exponent decreases) the weight of the sentiment increases since agents are trading based on the same set of information which consists of news and public mood. As we have seen that a single tweet might sometimes destabilise the market and change the price of cryptocurrencies that are heavily correlated. Both noise traders and even bigger whales are sensitive to public mood and news on different scales. Since ever large investors usually quit relatively risky assets in time of turbulence and economic crises. Among existing models, a stochastic volatility type model where the volatility is an indicator of sentiment calibrated on public mood data.

Chapter 5

An evolving risk management framework

"Businessmen need to understand the psychology of risk more than the mathematics of risk."

Paul Gibbons

5.1 Introduction

After studying the dynamics of Bitcoin and figuring out the model that we will use to price instruments derived on this new born asset. It is time to answer one of the main questions in this thesis. What risk does holding derivatives of bitcoin present to an institution ? and how much capital should we reserve when adding crypto assets as underlying in an investment portfolio ?

5.2 Prudential Treatment of Cryptoasset Exposures

5.2.1 Legal risk

Since the increase of interest in the new digital crypto market and all its effects on the financial sector. There has been an increasing need to clarify both governments and regulators point of view, and its legal implications, towards this new phenomena. Investors are trying to grasp the opportunity to make profits investing in this new market while also taking legal risk since central banks, regulators, associations and goverments are trying to understand the nature of digital currencies and its impact on the world's economy.

Cryptocurrency is unlike electronic money as it cannot be physically owned and transferred between agents. All legal risks come from the blurriness of the cryptocurrencies'

nature and how people should react to it. It is like human beings discovering fire for the first time. We don't know wether we should avoid it or embrace its potential. In some countries, crypto is considered as a currency which is the case of Salvador that legalized bitcoin on the 7^{th} of September, 2021. With the IMF urging it not to consider it as a currency, especially after the sudden burst in the end of January 2022. Salvador might end up prohibiting bitcoin and investors this way would lose a lot of money.
Added to that, this different understanding of cryptocurrency might cause a real legal risk to investors concerning how should it be taxed.
As we have seen in the first part of this thesis, bitcoin exhibits different behaviour that makes it hard for us to classify it in a specific market like equity, commodities... Some countries like the Salvador considers it to be a currency while others, like the United States, consider it to belong to the equity market. This creates a problem in the amount of taxation to be accounted for. Since in the US, cryptocurrencies are considered as a property by the IRS. When it comes to reporting earnings on their annual tax returns investors are indebted to capital gain tax. However it is unclear whether holdings should also face additional tax if bought on a foreign exchanges. Confusion also rises when filling out the report of Foreign Bank and Financial accounts if holdings exceed 10 thousand dollars however this rule is not clear to be applied to cryptocurrency to this day and creates a black hole of uncertainty for investors.

Added to that Bitcoin and other cryptos have a decentralized status. Hence, not backed by any central authority. This could result in legal complications as the value of cryptos is totally dependant on the demand and supply. This is true to all currencies. Therefore, without a central authority backing their values, investors may be left in the chaos. Also since there is no trusted financial institution to register transactions legal confusion might take place between parties and legal recourse can be difficult if not impossible to asses, because of the decentralization.

5.3 Capital treatment of cryptoassets

Regulating the cryptocurrency market ,institutions and banks' exposure to cryptoassets has been one of the Basel's committee evolving projects over the past few years.
Establishing clear and certain regulations is a must since several financial associations like the ISDA[1], the CDC[2], the CFMA[3] ... are demanding a clear point of view about crypto. They think that this limited exposure is neither desirable nor sustainable. The reason is that cryptoassets have several advantages that must be acquired within the regulatory perimeter.
For instance, the distributed ledger technology can improve financial services by making them faster and more secure: On the blockchain, transactions data can be saved at a speed

[1]International Swaps and Derivatives Association
[2]Chamber of Digital Commerce
[3]Construction Financial Management association

and transparency that helps create a more efficient collateral management and better counterparty, liquidity and settlement risk mitigation since transactions and recording of assets is simultaneous.

Banks' participation would greatly help regulatory landscape. Their vast expertise in identifying, monitoring, and continually managing risks. In exchange they would benefit from a risk-sensitive and adequately designed prudential framework, as it would provide a vast land for innovation in order to respond to customer demand. For this reason, the Basel committee must acknowledge hedging in the prudential framework.

5.3.1 Why should banks be involved in cryptoassets

As discussed earlier, the Prudential framework should encourage bank involvement in the development of safe and efficient cryptoasset markets, yet with too restrictive regulations bank involvement would be precluded. In this section we will discuss why we shouldn't keep the banks away from shaping the cryptocurrency market.
If banks are not in this process serving as intermediaries, unregulated agents will keep expanding and this will definitely lead to market fragmentation that translates to fragility in times of turbulence and stress due to the less transparent nature of an unregulated sector. There is, therefore, a need for a more empirical and analytical framework that extends to banks, leading to a more stable market as banks continuously identify, monitor and manage risks from both a prudential and behavioural perspective.

Here, we further develop the advantages of bank involvement:

- Integration of investment banks in the cryptocurrency market will increase transparency: Banks are always supervised and examined by several regulators in the whole world. As a matter of fact, they are obliged to provide transparent information and to report to supervisor allowing access to different information on periodic basis. As a result, activities conducted within a regulated bank are fully transparent to supervisors. So they can use information regarding that activity to inform about potential financial stability concerns.

- Another problem that might result from discarding banks from this market is its fragmentation. Agents definitely want to invest in crypto assets seeking billions of dollars products and services that have an effect on economies of scale from nonbank financial intermediaries. This would lead to a concentration of risk in unregulated financial sectors. Clients, therefore, won't be as much protected using regular banks services. Added to that we would have a major concern of money laundering and anonymous customers.

- One of the major advantages of DLT is it's transparency and ability to record transaction information the time it is taking place. Bank involvement in such a technology would help it to significantly reduce operational risk. Banks harnessing the distributed ledger technology for cryptoassets and other services could also benefit from a great efficiency to collateral management, trading and settlement. Faster processes will generally reduce counterparty and settlement risk and mitigate trades break down.

- Banks can bring their strong expertise in risk management to cryptoasset markets. Banking sector have known several crises, that made them experts at risk management and helped them develop their own complex models that will protect them from turbulent times in crypto market. Added to that they designed products that can maintain strict limits and ensure that customer activity is both traceable and reportable.

- Finally, Banks can reduce volatility in this new born market. They can play a pivotal role in ensuring required liquidity and transparency. A liquid market is generally a market with low volatility as this ensures an equilibrium between investors and avoids sudden movements coming from agents altering the market. Moreover the hedging tools that banks can offer will help agents mitigate risk and thus reduce volatility.

5.3.2 Prudential treatment of cryptoassets exposures: The Basel committee

In June 2021, the basel committee have issued its consultative document for the treatement of cryptoassets(on Banking Supervision, 2021). The committee have classified crypto-assets into two groups:

- Group 1: is composed of cryptoassets who are either tokenised standard financial asset or that are linked in value to an underlying traditional asset or a an index of traditional assets. Those assets are usually called stable coins or assets. Added to that their transaction are done by regulated institutions. The functions of the cryptoassets and its network, including DL, are designed to mitigate and manage material risks. Group 1 itself is therefore divided into two groups:

  - Group 1a, which are tokenised assets
  - Group 1b which are cryptoassets with effective stabilisation mechanisms like USDT.

  This group is subject to equivalent risk-based capital requirements based on the risk weights of underlying exposures as set out in the existing Basel capital framework.

- Group 2: Is composed of tokens that don't meet the definition of group 1. The basel committee think that they pose higher risks. However this classification might be extremely general since not all cryptocurrencies behave in the same way. There is a huge differnece between coins like Bitcoin or Etherium and others like Shiba coin. Since some coins have enough liquidity that can mitigate the risks.

Capital requirements

Group 1 Although we are far more interested in Group 2 crypto assets The Basel committee have set for Group 1 a richer proposal of regulation than of that of Bitcoin and other cryptocurrencies. This section describes the minimum risk-based capital requirements for credit and market risk for Group 1[4]. The most important points are:

- Activities related to cryptoassets will give rise to an operational risk charge within the Basel framework. Since theses cryptoassets technlogy is considered new the risks are unanticipated. Those risks could be managed by applying a Pillar 1 add-on operational risk charge for all Group 1 cryptoassets to which a bank is exposed. However the calibration of this charge is challenging and not yet defined.
- The committee set the minimum capital requirements for credit risk and market risk.
- Crypto-assets in Group 1 should be subject to the requirements set forth in the Basel framework for determining their allocation between the banking and trading books and for determining whether exposures are treated using standardized or internal model-based approaches. While internal model-based approaches are not prohibited under the treatment proposed in this consultation, supervisors should exercise great caution in deciding whether to allow such approaches, given the novel characteristics of crypto-assets.

Group 2 According to the Basel committee Group 2 cryptoassets are a lot riskier then Group. The requirements not only apply to all cryptos that don't satisfy the first classification but also to funds that are based on these assets like ETFs and derivatives, futures and products lying on them. The problem with the basel treatement is that is too simple and conservative for this category especially liquid and project based cryptos. The capital requirement consists of defining a risk weight of 1250% applied to the maximum of the absolute value of the all long positions and that of the short positions of the institution.

$$RWA = 12.5\, max(\|Short_{positions}\|\,, \|Long_{positions}\|) \tag{5.1}$$

The risk weight is chosen so that the bank is capitalizing all the exposure[5]. This RWA is computed independently for every cryptoasset exposure of the bank.
Another too conservative approach is that when the bank is holding a derivative the exposure isn't the derivative value yet the underlying exposure. The formula used is

$$min(underlying\, value, maximum\, loss)$$

Meaning if the underlying value exceeds the maximum loss, the maximum loss is taken instead. However this conservative approach includes both credit and market risk plus the

[4] The requirements only apply to those Group 1 cryptoassets which have not been deducted from Common Equity Tier 1 capital.

[5] Capitalisation=0.08*RWA where 0.08*12.5=1

counterparty risk (CVA[6] in this case).

In order to compute credit risk for derivative exposures: the Replacement Cost (RC) is added to the Potential Future Exposure (PFE), where the PFE is to be calculated as 50% of the gross notional amount. When calculating the RC, netting will be allowed within eligible and enforceable netting sets but not allowed between different cryptoassets.

Netting sets with a single counterparty that consists of only types of derivatives related to cryptoassets or derivatives related to cryptoassets and traditional asset transactions, would be treated separately.

The application of the 1250% risk weight set out in the above formula will ensure that banks are required to hold risk-based capital at least equal in value to their Group 2 cryptoasset exposures. In other words, the capital will be sufficient to absorb a full write-off of the cryptoasset exposures without exposing depositors and other senior creditors of the banks to a loss. The application of a 1250% risk weight to an asset is similar in effect to the deduction of the asset from capital. Unlike a deduction, however, a risk weight approach can also be applied to short positions, where there may be no balance sheet asset to deduct. For simplicity, the above formula also applies the 1250% risk weight to short positions. Theoretically, short positions and certain other types of exposures could lead to unlimited losses and thus, in some circumstances, the formula could require capital that is insufficient to cover potential future losses. Banks will be responsible for demonstrating the materiality of these risks under the supervisory review of cryptoassets and whether risks are materially underestimated. Supervisors will be responsible for considering an additional capital charge in the form of a Pillar 1 add-on in cases where banks have material exposures to short positions in cryptoassets or to cryptoasset derivatives that could give rise to losses that exceed the capital required by the 1250% risk weight. In applicable cases, the capital add-on would be calibrated by requiring banks to calculate aggregate capital requirements under the Committee's revised market risk framework (applying a 100% risk weight for delta, vega, and curvature) and Basic CVA risk framework (BA-CVA) and to use this amount if the result is higher than the requirement based on a 1250% risk weight.

[6]Credit valuation adjustment is an adjustement to the market value of derivative instruments to account for counterparty credit risk and accounts for the ability of a counterparty's to default.

5.3.3 Proposal for new regulations

We will focus in this part on Group 2 Assets which include Bitcoin. As stated in the different associations response to the Basel proposal, the classification of cryptocurrencies lacks granularity since Group 2 includes different kinds of coins that behave in a very different way. For example, the behaviour and risk for Etherium and Shiba coin is not the same. One of them has a sufficient market liquidity and a project that backs it up while the other is created out of a hype that might end in any moment and cause a permanent crash. Furthermore, the use of a single, punitive risk weight for Group 2 cryptoassets, rather than incorporating the existing risk treatments under the Basel framework, compromises longstanding prudential framework principles.

ISDA added to CDC and IIF[7] proposed that this group should be further devised into two groups (ISDA, 2021): Group 2a and Group 2b. Group 2a, for example, shall include cryptos for which there is a liquid two-way market. Although they exhibit volatile behaviour, they also exhibit market depth and price discovery. In addition, as discussed elsewhere in this response, the nature of banks' exposures to Group 2 cryptoassets may vary widely with materially different results in risk profiles. So the existing prudential framework for market risk and derivatives should be adjusted in order ro incorporate crypto assets, especially group 2a.

Capital Requirement

The single risk weight applied without considering any netting or hedging touls or benefits is too conservative. With crypto assets of Group 2a, there is enough liquidity and an evolving market that permits banks to use collateralization, hedging, and counterparty netting. That is why, if adjusted, the standards of the prudential framework can be applied to group 2a cryptoassets.

We can see that Group 2a exhibits correlation with their derivatives even in times of high volatility and market disturbance. This can demonstrate how well they are adapted to market risk hedging techniques. When applying a high risk weight to the greater of the absolute value of long and short positions, appropriate risk mitigation thanks to this correlation is avoided rather than used.

Group 2a could be held in the trading and banking book. However price exposures for this group is captured in a better way in the market risk framework. The FRTB[8], SA-CCR[9] and CVA framework should be applied too.

Prudential Framework adjustments

FRTB When adjusting the FRTB framework we would maintain simplicity of risk management, compute more accurate risk weighted assets inline with the risk and properties

[7]International Institute of Finance
[8]Fundamental Review of the Trading Book
[9]Standard Approach of Counterparty Credit Risk

of those assets. Cryptoassets should be added to the sensitivities based model. First of all if they are defined in the bank's reporting currency that would allow the separation of its risk from other traditional currencies risk. Added to that sensitivities for the same crypto asset should be netted before applications of risk weight.
As stated in the response of the financial associations to the the BIS proposal a risk weight of 90% to 95% should be applied based on price changes in a porfolio of group 2a cryptoassets composed of BTC, BTC cash and futures and ETH with a conservative liquidity horizon of 20 days from October 2017 to July 2021[10].
The analysis were first performed for 10 days horizon price and volatility and 20 days horizon price and volatility data, computing thus 99% two sided value at rik (VAR) and two sided 97.5% Expected shortfall (ES) also known as Conditional VaR.

Liquidity Horizon	Group 2a	
	99% VaR	**97.5**% ES
10 days	65%	68%
20 days	92%	95%

A similar computation were done using a Monte Carlo approach diffusing the market with the rough Heston model for a BTC, BTC vanillas, and Futures sample portfolio and calculating the VaR and CVaR[11] values from the different P&Ls got us the following results.

Liquidity Horizon	Group 2a	
	99% VaR	**97.5**% ES
10 days	71%	75%
20 days	96%	99%

The results differ since we didn't use the same portfolios neither the same model. The results show us that a general requirement applied for every crypto asset is rather not appropriate for those crypto assets that even belong to a new born market yet show consistency, project and market depth. That is why one should adjust the already existing framework to them in order to integrate banks in the process and further develop this market.
One must also say that correlation of Group 2a crypto assets can offer hedging opportunities and relatively independence from traditional markets can offer serious diversification for options and hedging possibilities.
The next of Mazars' project for the integration crypto assets is the adjustment of our Economic Scenario Generator and the integration of the implementation of these models in order to have a more general vue on the inclusion of crypto assets in the financial institution investments.

CVA The CVA risk of crypto derivatives is not very different of other assets or underliers. Banks should be able the already existing frame work with same weights, bucketing etc...

[10]A risk weight of 90% corresponds to 1125 % in RWA
[11]For definitions see chapter 3

However, as opposed to what the financial associations think. Volatility of cryptos with a lack of diversification might lead to a more probable default. That is why, some of these factors like the PD of the default process should be adjusted in order for the present framework to be efficient.

SA-CCR The counterparty credit risk for derivatives of crypto-assets is the same of traditional ones like equity, commodity or credit. However as stated before the exposure might be different but not the nature of the risk.

5.3.4 New regulations

The Basel committee issued a new proposal in June 2022. The basic structure is maintained as the main change concerns unbacked cryptoassets and stablecoins with ineffective stabilisation mechanisms also known as group 2 cryptoassets. As, as advised, a distinction has been made between project backed coins with sufficient liquidity and other coins. The updated proposal is no longer too conservative and propose adopted market risk tools with risk add-ons to group 2a cryptoassets. Given the fast evolution and nature of the cryptoasset market, the Committee will continue to monitor developments during the consultation periods with the help of financial associations.

Chapter 6

Conclusions and future recommendations

Cryptocurrency market is a gold mine for people investing in it. Large institutions like banks, hedge funds or insurance companies are showing so much interest in this emerging market as they want a piece of the cake of these phenomenal returns during the last ten years. However these returns come with a price: volatility and unpredictability of the price which can make those institutions insolvent in a blink of an eye if they cannot manage the cryptocurrency risks. In the scope of this project, we tried to understand this new-born market in order to hedge its multiple risks.

In the scope of the first chapter, we have studied the properties of the Bitcoin derivatives market to asses information that can be useful during the modelling process.

For the first part we started by several standard stochastic models, and we have shown at each time why one should add complexity to the model varying from stochastic volatility to jumps. The last model implemented was the Stochastic volatility model with correlated jumps, used by (Hou et al., 2020), that groups all characteristics of the previous models. This model showed relatively good results. Bitcoin volatility surface resembles the commodity surface as we can see a forward skew. We can also see that the behaviour of the surface really depends how investors view the market.

Finally we have shown that both the price and volatility processes of Bitcoin exhibit multi-fractality using the multi-fractal detrended fluctuation analysis and wavelets analysis. Those two methods showed that the bitcoin market, actually, follows the fractal market hypothesis theory and that the process presents persistence or roughness from time to time.

We considered a rough volatility model, the rough Heston model and calibrated it using neural networks. Moreover, we discussed the potential need for a multi-fractal model. These models are still not commonly used in quantitative finance due to their complexity but are rich enough to capture the changing dynamics of cryptocurrency.

Last but not least, we present the new proposal of regulations proposed by the Bale committee and the recommendations issued by several financial associations. For future work, one can look into some fundamental issues like the interest rate and transactions' fees used for bitcoin modelling, study how the technical characteristics of several crypto-currencies alter with its price dynamics. And finally build a capital friendly strategies using crypto-assets and options to test whether the regulations issued fit this market.

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Appendix A

MCMC Algorithms

Following definitions are extracted from (Palomba, 2019)

A.1 Gibbs Sampling

When iterative direct sampling from all of the complete conditionals is possible via standard methods, the resulting MCMC algorithm is a Gibbs sampler. In general, given $\left(\Theta^{(0)}, X^{(0)}\right)$, a Gibbs sampler is defined by

$$
\begin{aligned}
&\text{1. Draw } \Theta^{(1)} \sim p\left(\Theta \mid X^{(0)}, Y\right)\\
&\text{2. Draw } X^{(1)} \sim p\left(X \mid \Theta^{(1)}, Y\right)
\end{aligned}
$$

Iterating the two steps, the Gibbs sampler generates a sequence of random variables, $\left\{\Theta^{(g)}, X^{(g)}\right\}_{g=1}^{G}$, that converges to $p(\Theta, X \mid Y)$. The algorithm runs until it converges, and then a sample is drawn from the limiting distribution.

The Gibbs sampler requires that one can conveniently draw from the complete set of conditional distributions.

A.2 Metropolis-Hasting algorithm

[1] When one of the parameter posterior conditionals, namely $\pi\left(\Theta_i\right) := p\left(\Theta_i \mid \Theta_{(-i)}, X, Y\right)$, can be derived (as a function of Θ_i), but it is not possible to generate a sample from the distribution. Consider a single parameter and suppose we are trying to sample from a one-dimensional distribution, $\pi(\Theta)$, i.e., we are suppressing the dependence of other parameters and states in the conditional posterior, $p\left(\Theta_i \mid \Theta_{(i)}, X, Y\right)$. To generate samples from $\pi(\Theta)$, a Metropolis-Hastings algorithm requires the specification of a recognizable proposal or candidate density $q\left(\Theta^{(g+1)} \mid \Theta^{(g)}\right)$. This distribution will generally depend on the other parameters, the state variables and the previous draws for the parameter being

[1] The basics of this algorithm is derived from (Johannes & Polson, 2010)

drawn. As in Metropolis, et al., we only require that we can easily evaluate density ratio $\pi\left(\Theta^{(g+1)}\right)/\pi\left(\Theta^{(g)}\right)$ and this assumption is satisfied in the majority of continuous-time models.

The Metropolis-Hastings algorithm samples iteratively as in the Gibbs sampler, but it first draws a candidate point that will be accepted or rejected based on the acceptance probability. The Metropolis-Hastings algorithm consists of the following two stage procedure:

Step 1: Draw $\Theta_i^{(g+1)}$ from the proposal density $q\left(\Theta_i^{(g+1)} \mid \Theta_i^{(g)}\right)$

Step 2: Accept $\Theta_i^{(g+1)}$ with probability $\alpha\left(\Theta_i^{(g+1)}, \Theta_i^{(g)}\right)$ where

$$\alpha\left(\Theta_i^{(g+1)}, \Theta_i^{(g)}\right) = \min\left(\frac{\pi\left(\Theta_i^{(g+1)}\right)/q\left(\Theta_i^{(g+1)} \mid \Theta_i^{(g)}\right)}{\pi\left(\Theta_i^{(g)}\right)/q\left(\Theta_i^{(g)} \mid \Theta_i^{(g+1)}\right)}, 1\right)$$

Specifically, implementing Metropolis-Hastings requires: drawing a candidate $\hat{\Theta}_i$ from $q\left(\Theta_i \mid \Theta_i^{(g)}\right)$, drawing $u \sim$ Uniform $[0,1]$, accepting the draw, that is, set $\Theta_i^{(g+1)} = \hat{\Theta}_i$ if $u < \alpha\left(\Theta_i^{(g)}, \Theta_i^{(g+1)}\right)$, and otherwise rejecting the draw, that is, set $\Theta_i^{(g+1)} = \Theta_i^{(g)}$. This algorithm splits the conditional distribution into two parts: a recognizable distribution to generate candidate points and an unrecognizable part from which the acceptance criteria arise. The acceptance criterion insures that the algorithm has the correct equilibrium distribution. Continuing in this manner, the algorithm generates samples $\left\{\Theta^{(g)}\right\}_{g=1}^{G}$ whose limiting distribution is $\pi(\Theta)$.
It is then straightforward to see that Gibbs sampling is a special case of Metropolis-Hastings, where $q\left(\Theta^{(g+1)} \mid \Theta^{(g)}\right) \propto \pi\left(\Theta^{(g+1)}\right)$. This implies that the acceptance probability is always one and the algorithm always moves. The Metropolis-Hastings algorithm allows the functional form of the density to be non-analytic. When there are constraints in the parameter space, one can just reject these draws. In addition, sampling can be done conditional on specific regions, providing a convenient approach for analyzing parameter restrictions imposed by economic models.
The problem with this algorithm is that the choice of proposal density can affect the performance of the algorithm as in some cases, the algorithm may never converge, getting stuck in a region of the parameter space.

There are two important special cases of the general Metropolis-Hastings algorithm which deserve special attention.

- Metropolis-Hastings algorithm can draw $\Theta^{(g+1)}$ directly from proposal density, $q\left(\Theta^{(g+1)} \mid \Theta^{(g)}\right)$, which has a dependence from the previous Markov state $\Theta^{(g)}$ (and, in general, other parameters and states) or from a distribution independent of the previous state, $q\left(\Theta^{(g+1)} \mid \Theta^{(g)}\right) = q\left(\Theta^{(g+1)}\right)$. The second is known as an independence Metropolis-

Hastings algorithm:

Step 1: Draw $\Theta_i^{(g+1)}$ from the proposal density $q\left(\Theta_i^{(g+1)}\right)$

Step 2: Accept $\Theta_i^{(g+1)}$ with probability $\alpha\left(\Theta_i^{(g+1)}, \Theta_i^{(g)}\right)$ where

$$\alpha\left(\Theta_i^{(g+1)}, \Theta_i^{(g)}\right) = \min\left(\frac{\pi\left(\Theta_i^{(g+1)}\right)/q\left(\Theta_i^{(g+1)}\right)}{\pi\left(\Theta_i^{(g)}\right)/q\left(\Theta_i^{(g)}\right)}, 1\right)$$

The candidate draws, $\Theta^{(g+1)}$, are then drawn independently from the previous state, but in general the sequence $\left\{\Theta^{(g)}\right\}_{g=1}^{G}$ will not be independent while the acceptance probability depends on previous draws. When using independence Metropolis, it is common to pick the proposal density to closely match certain properties of the target distribution.

- The original algorithm considered by Metropolis, et al.is the so called Random-walk Metropolis. It draws a candidate from $\Theta^{(g+1)} = \Theta^{(g)} + \varepsilon_t$, where ε_t is an independent mean zero error term (e.g. a symmetric density function with fat tails, like a t-distribution). The choice of the proposal density is generic, ignoring the structural features of the target density and the symmetry in the proposal density, $q\left(\Theta^{(g+1)} \mid \Theta^{(g)}\right) = q\left(\Theta^{(g)} \mid \Theta^{(g+1)}\right)$, leads to a simplification of the algorithm:

  Step 1: Draw $\Theta_i^{(g+1)}$ from the proposal density $q\left(\Theta_i^{(g+1)} \mid \Theta_i^{g}\right)$

  Step 2: Accept $\Theta_i^{(g+1)}$ with probability $\alpha\left(\Theta_i^{(g+1)}, \Theta_i^{(g)}\right)$ where

$$\alpha\left(\Theta_i^{(g+1)}, \Theta_i^{(g)}\right) = \min\left(\frac{\pi\left(\Theta_i^{(g+1)}\right)}{\pi\left(\Theta_i^{(g)}\right)}, 1\right)$$

  The variance of the error term is under control and adjusted to tune the algorithm helping us obtain an acceptable level of accepted draws ($20\,to\,40\%$).

Appendix B

Crude Monte Carlo

Algorithm (Crude Monte Carlo for Independent Data)

1. Generate $Y_1, \dots, Y_N \overset{\text{iid}}{\sim} f$ (for example, from independent simulation runs).
2. Calculate the point estimate $\bar{Y}$ and confidence interval (13) of $\ell = \mathbb{E}Y$.

It is often the case that the output Y is a function of some underlying random vector or stochastic process; that is, $Y = H(\mathbf{X})$, where H is a real-valued performance function and $\mathbf{X}$ is a random vector or process. The beauty of Monte Carlo for estimation is that the TCL formula holds regardless of the dimension of $\mathbf{X}$.

Appendix C

Neural network

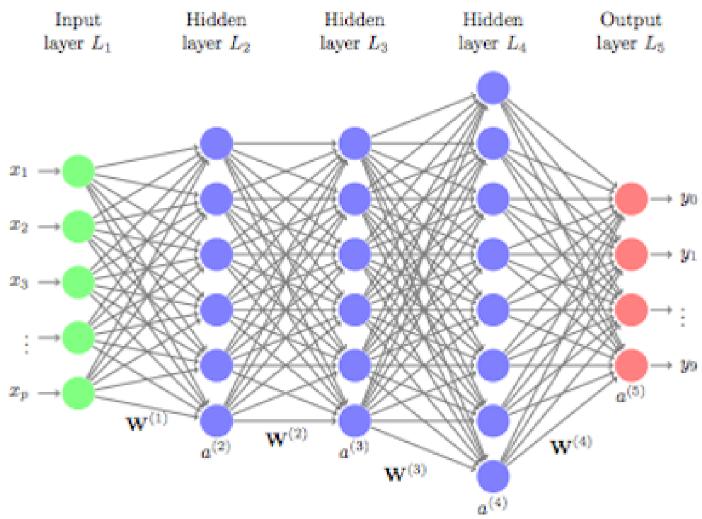

Figure C.1: a feed-forward Network

A feed-forward network is a mapping $F_{\text{net}} : \mathbb{R}^N \to \mathbb{R}^M$ being defined by its general architecture and a number of network weights denoted w. To evaluate F_{net} one then traverses from left to right in the graph and at each node (excluding the input layer) performs a computation of the form $y \mapsto \sigma(a'x + b)$ where y is the output from the last layer (a column vector), σ a non-linear (activation) function, a a coefficient (column-)vector and b a scalar. Letting w denote the collection of all such coefficients a and b, the problem of finding a neural network approximation to F then consists of finding both an architecture as well as weights w, that minimize the error between F_{net} and F. Assuming the architecture has already been fixed, this is typically done as follows: To start, one generates a synthetic (training) dataset of input-output pairs by evaluating F at various inputs x and storing the results $y = F(x)$. The inputs will in practise often be randomly sampled from some distribution covering the domain over which we wish to approximate F. Letting X be such a random input sample, we then define the generalisation error as the number

$$E\left[\mathcal{L}\left(F_{\text{net}}\left(X;w\right),F(X)\right)\right]$$

where $\mathcal{L}:\mathbb{R}^M\times\mathbb{R}^M\to\mathbb{R}$ is some loss function and $F_{\text{net}}\left(X;w\right)$ refers to the neural network being evaluated in X using weights w. An empirical version of (8) is then easily constructed on the training data by averaging errors across the samples. To train the network one can then minimize this empirical version with respect to w. This is usually done using stochastic gradient descent which is a gradient based optimization method where one iterates in epochs (i.e. cycles) across the training data, first shuffling all samples, then looping over smaller batches of the samples (covering all of them) each time updating the weights using a gradient estimate. Let us end by pointing out that the approximations can in fact be theoretically justified. As an example, the Universal Approximation Theorem of essentially says that any continuous function can be arbitrarily well approximated on a compact domain by a single hidden-layer network. It shall though be noted, that it generally is considered true that deep neural networks have an improved approximation capacity.

Chapter 7

Executive summary

7.1 Introduction

The history of crypto-assets can be traced back to the creation of Bitcoin by "Satoshi Nakamoto" in 2008. Since then, the blockchain technology and cryptocurrencies have gained much interest for many investors. The emergence of Bitcoin options and futures in cryptocurrency derivatives exchanges have announced the beginning of a new epoch in Bitcoin price risk hedging. Options and futures give a certain freedom to trade and hedge volatile fluctuations in the asset price effectively. As opposed to other financial markets, a little research was done on this growing market which presents a major hurdle for institutions that want to invest or already investing in bitcoin like Grayscale, Ark Invest, Tesla ... In the scope of this report, we investigate crypto-assets dynamics to better hedge the risk of investing in this new-born market. Our approach consists of using stochastic models to better model bitcoin as long as other crypto-assets. We finally review the new proposal for regulation and the different recommendations of the financial associations.

7.1.1 The cryptocurency market

Bitcoin Markets' Actors The bitcoin markets' actors are **the miners** who are crucial participants that verify the legitimacy of Bitcoin transactions through the Proof-of-Work and are rewarded in bitcoin, **Retail Investors and Traders**, **Institutional Investors and Traders** who significantly increased their adoption in recent years, with hedge funds and quant traders from major investment banks (JP Morgan, Goldman Sachs,etc..) leading the way. **Corporations** and **Exchanges** who act as a market intermediate between two parties. Those who want to buy and those who want to sell. Their fees are minimal compared to other markets however they have control over the cryptocurrency which can lead to high security risks. Among the most important exchanges the Chicago CME, Coinbase, Deribit from which we got our options data which consists of European style vanilla options. Fees are neglected since they are minimal.

Bitcoin's price Bitcoin is actually scarce meaning that there will never be more than 21 million bitcoin. It is immune to quantitative easing and other inflationary measures and can be easier to transport, store, and divide. The price is determined through supply and demand. Bitcoin is traded 24 hours a day 7 days a week and therefore there is no closing price. Its value is determined upon a rolling average. Finally, the level of supply and demand may vary across different exchanges. Hence, the difference observed in Bitcoin price.It's hard for traders to arbitrage differences across exchanges because of the high cross fees.

Liquidity: spread and volume Liquidity explains how easy a particular asset can be exchanged without altering the stability of its price. We can see in fig 2.3 that 50-days average trading volume for bitcoin options is 639 Mil dollars which represents nothing compared to the SP500 where the 50-days options average trading volume exceeds 60 billion dollars. Greater trading volume means greater trading activity and is therefore a strong indicator of a liquid market. The bid-ask spread is the difference between the price in the order book at which market agents are willing to sell or buy a derivative. As we can see in fig 2.5 in the report the spread for a call option almost at the money is almost 1300 *bp* which is considered to be enormous compared to the For-ex market where the spread of EUR/USD doesn't exceed 5 *bp*. If a number represents a valid security price, then there must be parties in the market willing to transact the security at that price. To improve price accuracy we may want to increase order size. However, at some point the order will become so large that it will affect price. The spread decreases yet the price is distorted. This quality has been pointed out to resemble the Heisenberg's uncertainty principle in quantum mechanics.

Transaction Costs It is true that when trading short term, transaction costs can make a winning strategy a losing one. However, in this project, we decided to neglect transaction costs to simplify the modeling process.

7.2 Cryptocurrency market modelling

To better understand the market. We started exploring some of the most known models adding some complexity to it on the run, if needed.

7.2.1 Black & Scholes

Assumptions Black-Scholes model assert some strong assumptions like the completeness and efficiency of the market and that the returns of the underlying are Gaussian, stationary and independent. The interest rate used is the risk-free rate for the USD and transaction costs have been neglected to simplify the modelling.

The mathematical model and the volatility surface The price process in the Black-Scholes model follows this dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \text{ for } \mu, \sigma > 0 \tag{7.1}$$

Using the Ito formula we can deduce the price function of vanilla C_{BS} options and other derivatives. The real purpose of this model is to investigate the implied volatility of bitcoin's options. Giving that the vanilla options price function in the Black-Scholes model is bijective, we can inverse it to get the volatility from market prices. To solve the latter inversion problem two numerical methods were used: The Newton-Ralphson and bisection methods. Yet Newton-Ralphson algorithm was faster to converge as it needed no more than 5 or 6 iterations. One can see the different surfaces interpolated using cubic spline method in fig 3.1 in the report.

Interpretation We observe the same features in a standard market. The smile is caused by several reasons. The most important one is probably the supply and demand. For instance in the FX market the smile is symmetrical. We can see a shade of this in 3.1 on the 17^{th} of august. For bitcoin the reason behind this is that many investors use bitcoin options equally for hedging risk and for speculation, so they might sometimes be interested in both ITM calls and OTM puts. Looking at the same figure but on the 30^{th} of July, the volatility surface exhibits a heavy skew just like the equity market and that is due to the fact that investors have to protect themselves against large drops since Bitcoin price knew a huge drop in the end of July specifically so they get protective puts to hedge that risk. For the second data set. The forward volatility skew is fairly apparent. This suggests that the demand for buying out-of-the-money calls and in-the-money puts has increased significantly not only hedge the Bitcoin price risk yet to speculate its increase. Eventually, as the options approach maturity, the implied volatilities rose to more than 180% for some trading day. Therefore, the appropriate interpretation of the increase in implied volatilities is the demand for these strikes.

Conclusion By looking at the properties of the bitcoin volatility smile, the existence of forward volatility skewness resembles the skewness of traditional commodity markets rather than equity ones. One can conclude that Bitcoin might belong to the commodity class of assets. The use of the BS model has shown that it is necessary to add jumps to capture sudden changes in the price and to try the stochastic model in order to capture the dynamics of the volatility.

7.2.2 Merton model

Assumptions The model is based on the assumptions of the Black-Scholes model. The interest rate used is the risk-free rate for USD and transaction costs have been neglected to simplify the modelling process.

The Mathematical model The dynamics of the price process is given by:

$$\begin{cases} dX_t = \mu dt + \sigma dW_t + JdN_t \\ h\left(X_t\right) = \exp\left(X_t\right) = S_t \end{cases}$$

where σ and μ are constants and the jump sizes are normal $\forall i, J_i \sim \mathcal{N}\left(m, \delta^2\right)$. Therefore let be $Q_t = \sum_{i=1}^{N_t} J_i$ a compound Poisson process describing the jump process. For vanilla options we have explicit price formulas which makes the calibration easier.
We need to calibrate the four parameters of the model with are the volatility σ, the jumps occurrence parameter λ, the jumps size mean and variance m and δ^2. The process is done through a simple optimization problem solved by the Sequential Least Squared Quadratic Programming (SLSQP) developed by (Kraft, 1988) already implemented in *Python*'s package `scipy`. Results are shown in 3.2.

Results and conclusions The Merton jump diffusion model was able to price the vanilla options with an relative mean squared error of 14.89 %. When compared to the Black-Scholes model The Merton model was able to reproduce in a way the volatility smile as we see in 3.3. The strength of jump diffusion processes is that it fits short-term skews that are usually more pronounced since adding jumps captures sudden variations in close future. However, volatility is constant over time. A Heston model will then be used to capture the dynamics of volatility.

7.2.3 Stochastic volatility

The mathematical model

Let S_t be the the price process and V_t be the volatility process. The dynamics of the Heston model are described as follows.

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{(S)} \\ dV_t &= \kappa\left(\bar{v} - V_t\right) dt + \sigma \sqrt{V_t} dW_t^{(V)} \\ d\left\langle W_t^{(S)}, W_t^{(V)} \right\rangle &= \rho dt \end{aligned} \tag{7.2}$$

The drift term in the volatility dynamics is mean reverting when $\kappa > 0$, with $\bar{v}$ being the long-term mean level of the variance. In fact, if at time t the process V_t is greater than θ, the drift term will push the process value down.

Calibration of the model

The calibration of the Heston model is trickier than the other models. First of all there are 5 parameters that need to be calibrated and the objective function is not known to be convex. A complexity is added later on due to the dependency of some parameters. As a matter of fact σ and κ offset each other. That might be due to the fact that the objective function is flat reaching the optimum.

Like the other models we will try to minimise a vanilla option's price and the one found on the market. We use (Cui et al., 2016) equations for the calibration process, as he simplified the optimisation problem. To solve the optimization problem we used the Levenberg-Marquardt(LM) mtheod. The LM algrithm is a typical tool to solve a nonlinear least squares problem, The search step is given by: $\Delta\boldsymbol{\theta} = \left(\boldsymbol{J}\boldsymbol{J}^{\top} + \mu\boldsymbol{I}\right)^{-1}\boldsymbol{\nabla} f$ By adjusting μ, the method changes between the gradient descent method and the Gauss-Newton method. lAbility of the model to capture forward skew

Results and conclusions Looking at the correlation between stock price and volatility, we can see that, as opposed to many empirical studies that have documented a negative correlation between the stock price and volatility processes (Black, 1976). In the case of bitcoin the correlation parameter ρ is positive since there is a higher deep OTM calls for speculation. The positive parameter might also indicate the presence of the noise trading behaviour as the rise of volatility attracts short term traders and thus the price increases. We used the model to price our options, overall the Heston model performed better than the ones before it. The relative mean squared error of the stochastic volatility model is 11.84% for July the 30^{th} and 9.76% for August the 18^{th}.
Using the Heston prices we derived the implied volatility surface in fig 3.6. Overall the model reproduces the volatility surface yet it performs poorly for short maturities. To solve this problem one can add jumps to a stochastic volatility model.

7.2.4 Stochastic volatility with co-jumps

The mathematical model

Let S_t be the price process and V_t the volatility one, the SVCJ dynamics are as follows:

$$\begin{gathered} d\log S_t = \mu dt + \sqrt{V_t} dW_t^{(S)} + Z_t^y dN_t \\ dV_t = \kappa\left(\theta - V_t\right) dt + \sigma_V \sqrt{V_t} dW_t^{(V)} + Z_t^v dN_t \\ d\left\langle W_t^{(S)}, W_t^{(V)}\right\rangle = \rho dt \\ \mathbb{P}\left(dN_t = 1\right) = \lambda dt \end{gathered} \tag{7.3}$$

N_t is a pure jump process with a constant mean jump-arrival rate λ. The random jump sizes are Z_t^v, Z_t^y. We assume that the random jump size Z_t^y conditional on Z_t^v is

$$Z_t^y \mid Z_t^v \sim \mathrm{N}\left(\mu_y + \rho_j Z_t^v, \sigma_y^2\right); \quad Z_t^v \sim \exp\left(\mu_v\right)$$

Results and conclusions Following (Hou et al., 2020) the calibration is done using Markov chain Monte Carlo methods (MCMC). The calibration process yielded the parameters in table 3.1. The SVCJ model fits the data well and the significance of the jump parameters relatively explains the need of this model. The pricing of the cryptocurrency vanilla options is done using Crude Monte Carlo simulations[1] using: $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}C(T) \mid \mathcal{F}_t\right]$

[1]see appendix B

As expected the SVCJ model outperformed the previous models with an RMSE of 8.12 %. We were able to reproduce the SVCJ volatility surface in fig 3.8. We can see that this model represents well the volatility surface plus we can observe thanks to the error surface that we relatively solved the short-term skew fitting problem of the Heston model. The price of Bitcoin is on some level governed by noise trading. This makes us think that it might have a fractal behavior. In the next section, we investigate this hypothesis.

7.3 Cryptocurrency: A rough model in needed

Cryptocurrency and the fractal market hypothesis (FMH) The Efficient Market Hypothesis implies that the price process reflects all available information and therefore it is impossible for markets participants to consistently outperform the market on risk adjusted basis. In his book, Peters outlined a new theory based on empirical studies of the different markets: The fractal market hypothesis (FMH). This theory states that the market contains many investors with different investment horizons and different information sets. Investors with longer-term horizons base their decisions on fundamental information, yet shorter-term ones rely on technical aspects. This structure provides liquidity to the different agents making the market stable. During turbulent times the investment horizons and information sets become uniform which dries liquidity and therefore drives volatility upwards and therefore attracts more short term investors and noise traders.

Evidence of cryptocurrencies fractal behaviour

To capture the fractal structure of the market we use the multi fractal analysis and the continuous wavelet analysis.
The MF-DFA analysis is used to measure the general Hurst exponent $h(q)$. For $q = 2$, $h(2)$ corresponds to the well-known Hurst exponent H If $h(2) < 0.5$, the time series is anti-persist and if $h(2) > 0.5$, it is persistent. For $h(2) = 0.5$, the time series becomes a random walk. Using this method for the daily volatility and price of bitcoin we plotted 4.1. We can see a multi-fractality behaviour. For the price data H is equal to 0.56. This means that curve of Bitcoin price shows persistence in general. Taking a look on the 1-min data, we notice a coefficient of 0.3 that shows a certain mean-reversion and a turbulent state of the market for the last 10 days of September. When investigating the price process during this period of time we can see a mean reversion character and a slight increase in the historical volatility.
When applying the same analysis to the yearly data. We notice that for every year a multi-fractal behaviour is captured. We can also see that some years exhibit anti-persistence. For example, the last year, the returns process shows roughness which is consistent with the large prices changes that bitcoin knew and its sensitivity to external events and market news which implies the dominance of short-term trading horizons. A variable Hurst parameter, fig 4.3, also suggests that the dynamics of bitcoins price vary overtime and a more general model should be used to capture this variable behaviour. Applying the

Multi-fractal analysis to the historical volatility, we see that it exhibits also a multifractal behaviour. The Hurst exponent is less than $\frac{1}{2}$ which is consistent with results empirically observed for other assets. This results supports the use of a rough volatility model.

Wavelet analysis

The strength of the wavelet analysis is its ability to visualize the underlying information both frequency and time domains, thus providing information about the price evolution across different frequency and time scales. According to the FMH arguments, we should observe increased power at low scales or high frequency during the critical periods. Seeing fig 4.4 we can confirm those hypothesises. Moreover, we observe a changing structure of variance across frequencies before the turbulences due to the changing structure of investors' activity. Important and significant power (volatility) is detected at high frequencies confirming, thus, the dominance of high frequency trading.

7.4 Rough volatility model

Due to the fractal behaviour spotted in the previous part. We used the rough Heston model described as follows:

$$\begin{gathered} d\log S_t = \mu dt + \sqrt{V_t} dW_t^{(S)} \\ V_t = \xi_0(t) + \frac{\nu}{\Gamma\left(H+\frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}} \sqrt{V_s} dW_s^{(V)} \\ \operatorname{Cov}\left(dW_t^{(S)}, dW_t^{(V)}\right) = \rho dt \end{gathered} \tag{7.4}$$

where $\nu > 0, H \in \left(0, \frac{1}{2}\right)$. The rough character is due to the the kernel $(t-s)^{H-\frac{1}{2}}$. Let be $\alpha = H + \frac{1}{2}$.

And where $\xi_0(t) = E\left[V_t\right], t \geq 0$, which means the initial forward variance curve is an input. We began by simulating the price and volatility model using various techniques of fractal Brownian motion simulation. This helped us create an image of a market powered using this model and thus create volatility surfaces that would be used to train a Feed-forward neural network. Using this trained Neural Network, we were able to calibrate our model. The calibration yielded different parameters that were inline with the other models and helped us better the pricing accuracy by decreasing the RMSE to 3,78%.
Due to the multi-fractality behaviour spotted in the last analysis, one might try to use a multi-fractal model in the future to incorporate the changing dynamics of crypto-currency.

7.5 An evolving risk management framework

June 2021, the Basel committee have issued a new proposal of regulation for the crypro-currency market. This proposal was rich for tokenized assets and cryptocurrencies with efficient stabilization system yet was to restrictive for other cryptoassets including Bitcoin and Etherium and would result in non involving banks and financial institutions in this emerging market. A proposal including several improvements has been presented by the financial associations. Those improvements concerned group 2 cryptoassets that included Bitcoin and Coins with sufficient liquidity. Using VaR and CVaR measures we were able to confirm that the proposal of ISDA and other institutions regarding the prudential treatements of cryptoassets actually made sense.

7.6 Conclusion and future work

Cryptocurrency market is a gold mine for people investing in it. Large institutions like banks, hedge funds or insurance companies are showing so much interest in this emerging market as they want a piece of the cake of these phenomenal returns during the last ten years. However these returns come with a price: volatility and unpredictability of the price which can make those institutions insolvent in a blink of an eye if they cannot manage the cryptocurrency risks. In the scope of this project, we try to understand this new-born market in order to hedge its multiple risks. We showed that complex models that include stochastic volatility and jumps are needed to fit the volatility surface. We also showed that the market exhibits a fractal behaviour and therefore a rough stochastic model should be used to better model the underlying price and variance. Studying the current regulations, we can see that is too restrictive and can't actually help banks get involved in cryptocurrency in order for them to bring some order to this market. Using our last model, we were able to confirm the need for new regulation as proposed by the financial associations in September 2021. We can also confirm that the basel committee is providing new less restrictive regulations while trying to monitor cryptocurrecies risk. Vast improvements are being done to integrate this emerging market however the work is not yet done.

Chapter 8

Notes de synthèse

8.1 Introduction

L'histoire des crypto-actifs remonte à la création du bitcoin par "Satoshi Nakamoto" en 2008. Depuis, la technologie blockchain et les crypto-monnaies ont suscité l'intérêt de nombreux investisseurs. L'apparition d'options et de contrats à terme sur le bitcoin dans les marchés de produits dérivés sur les crypto-monnaies a annoncé le début d'une nouvelle époque dans la couverture du risque lié au prix du bitcoin. Les options et les contrats à terme offrent une certaine liberté pour négocier et couvrir efficacement les fluctuations volatiles du prix de l'actif. Contrairement à d'autres marchés financiers, peu de recherches ont été faites sur ce marché en pleine croissance, ce qui représente un obstacle majeur pour les institutions qui veulent investir ou qui investissent déjà dans le bitcoin comme Grayscale, Ark Invest, Tesla.... Dans le cadre de ce mémoire, nous étudions la dynamique des crypto-actifs afin de mieux couvrir le risque d'investir dans ce marché. Nous allons calibrer différents modèles stochastiques à l'aide des prix des options du marché afin de mieux pricer les instruments sur le Bitcoin et comprendre sa dynamique.

8.2 Le marché des crypto-monnaies

Acteurs du marché des crypto-monnaies Les acteurs du marché des bitcoins sont: **les mineurs** qui sont des participants cruciaux qui vérifient la validité des transactions en bitcoins par le biais de la "proof-of-work" et sont récompensés en bitcoins, **les investisseurs et les traders de détail**, **les investisseurs et les traders institutionnels** qui ont considérablement augmenté leur adoption ces dernières années, les hedge funds et les quant traders des grandes banques d'investissement (JP Morgan, Goldman Sachs, etc.) menant la danse. **Corporations** et **Bourses** qui agissent comme un intermédiaire de marché entre deux parties. Celles qui veulent acheter et celles qui veulent vendre. Leurs frais sont minimes par rapport à d'autres marchés, mais elles contrôlent les crypto-monnaies, ce qui peut entraîner des risques de sécurité élevés. Parmi les bourses les plus importantes, on trouve le Chicago CME, Coinbase, Deribit, d'où proviennent nos données sur les options, qui sont des options vanilles de style européen. Les frais sont négligés car ils sont minimes.

Le prix du bitcoin Le bitcoin est en fait rare, ce qui signifie qu'il n'y aura jamais plus de 21 millions de bitcoins. Il est immunisé contre les mesures d'assouplissement quantitatif et autres mesures inflationnistes et peut être plus facile à transporter, à stocker et à diviser. Le prix est déterminé par l'offre et la demande. Le bitcoin est échangé 24 heures sur 24, 7 jours sur 7, et il n'y a donc pas de cours de clôture. Sa valeur est déterminée par une moyenne mobile. Enfin, le niveau de l'offre et de la demande peut varier d'un exchange à l'autre. Il est difficile pour les traders d'arbitrer les différences entre les différents exchanges en raison des frais élevés.

Liquidité : Écart bid-ask et volume La liquidité explique la facilité avec laquelle un actif particulier peut être échangé sans altérer la stabilité de son prix. Nous pouvons voir dans la fig 2.3 que le volume d'échange moyen sur 50 jours pour les options sur bitcoin est de 639 millions de dollars, ce qui ne représente rien comparé au SP500 où le volume d'échange moyen sur 50 jours dépasse 60 milliards de dollars.
Un volume d'échange plus important signifie une plus grande activité commerciale et est donc un indicateur fort d'un marché liquide. L'écart bid-ask est la différence entre les prix du carnet d'ordres auxquels les agents du marché sont prêts à vendre ou à acheter un produit dérivé. Comme nous pouvons le voir dans la fig 2.5 du rapport, l'écart pour une option d'achat presque à la monnaie est de près de 1300 ; *bp*, ce qui est considéré comme énorme par rapport au marché For-ex où l'écart de l'EUR/USD ne dépasse pas 5 ; *bp*. Si un nombre représente un prix valide pour un titre, alors il doit y avoir des parties sur le marché prêtes à négocier le titre à ce prix. Pour améliorer la précision des prix, nous pouvons vouloir augmenter la taille des ordres. Cependant, à un moment donné, l'ordre deviendra si important qu'il affectera le prix. L'écart dimunue alors que le prix est faussé. Cette qualité a été soulignée pour ressembler au principe d'incertitude d'Heisenberg en mécanique quantique.

Les frais de transaction Lorsque on fait du trading à court therme, les frais de transaction peuvent rendre une stratégie gagnante à une stratégie à rendement négatif. Par-contre, dans ce projet, nous avons décidé de négliger les frais de transaction pour simplifier le processus de modélisation.

8.3 Modélisation du marché des crypto-monnaies

Pour mieux comprendre le marché. Nous avons commencé à explorer certains des modèles les plus connus en les complexifiant au fur et à mesure si nécessaire.

8.3.1 Black & Scholes

Hypothèses Le modèle Black-Scholes repose sur des hypothèses fortes comme l'efficience du marché et le fait que les rendements du sous-jacent sont gaussiens, stationnaires et indépendants. Le taux d'intérêt utilisé est le taux sans risque pour le USD et les frais de transactions ont été négligé pour simplifier la modélisation.

Le modèle mathématique et la surface de volatilité Le processus de prix dans le modèle de Black et Scholes suit cette dynamique :

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \text{ for } \mu, \sigma > 0 \tag{8.1}$$

En utilisant la formule d'Ito, nous pouvons déduire la fonction de prix des options vanille C_{BS} et d'autres dérivés. Le véritable objectif de ce modèle est d'étudier la volatilité implicite des options du bitcoin. Étant donné que la fonction de prix des options vanille dans le modèle de Black et Scholes est bijective, nous pouvons l'inverser pour obtenir la volatilité à partir des prix du marché. Pour résoudre ce dernier problème d'inversion, deux méthodes numériques ont été utilisées : La méthode de Newton-Ralphson et la méthode de bisection. Cependant, l'algorithme de Newton-Ralphson a été plus rapide à converger puisqu'il n'a pas nécessité plus de 5 ou 6 itérations. On peut voir les différentes surfaces interpolées à l'aide de la méthode des splines cubiques dans la fig. 3.1 dans le rapport.

Interprétation Nous observons les mêmes caractéristiques dans un marché standard. Le smile est dû à plusieurs raisons. La plus importante est probablement l'offre et la demande. Par exemple, sur le marché des changes, le smile est symétrique. Nous pouvons en voir une nuance dans 3.1 sur le 17 août. Pour le bitcoin, cela s'explique par le fait que de nombreux investisseurs utilisent les options sur bitcoin à la fois pour couvrir le risque et pour spéculer, de sorte qu'ils peuvent parfois être intéressés à la fois par des calls ITM et des puts OTM. En regardant la même figure mais en 30 juillet, la surface de volatilité présente un fort skew tout comme le marché equity et cela est dû au fait que les investisseurs doivent se protéger contre les grandes baisses puisque le prix du bitcoin a connu une énorme chute à la fin du mois de juillet spécifiquement, ils obtiennent donc des options de vente protectrices pour couvrir ce risque. Pour le deuxième ensemble de données. L'asymétrie de la volatilité implicite est assez apparente. Cela suggère que la demande d'achat d'options d'achat hors de la monnaie et d'options de vente dans la monnaie a considérablement augmenté, non seulement pour couvrir le risque lié au prix du bitcoin, mais aussi pour spéculer sur son augmentation. Finalement, à l'approche de l'échéance des options, les volatilités implicites ont augmenté à plus de 180 % pour certains jours. Par conséquent, l'interprétation appropriée de l'augmentation des volatilités implicites est la demande pour ces strikes..

Conclusion En examinant les propriétés du smile de volatilité du bitcoin, l'existence d'une asymétrie de la volatilité implicite ressemble à l'asymétrie des marchés de commodités classiques plutôt qu'à celle des marchés equity. On peut en conclure que le Bitcoin pourrait appartenir à la classe commodité. L'utilisation du modèle BS a montré qu'il faut ajouter des sauts pour capter les variations brusques du pris et de rendre la volatilité stochastique pour pouvoir capter la dynamique de la volatilité.

8.3.2 Le modèle de Merton

Hypothèses Le modèle repose sur les hypothèses du modèle du Black-Scholes. Le taux d'intérêt utilisé est le taux sans risque pour le USD et les frais de transactions ont été négligé pour simplifier la modélisation.

Le modèle mathématique La dynamique du processus de prix est donnée par :

$$\begin{cases} dX_t = \mu dt + \sigma dW_t + J dN_t \\ h(X_t) = \exp(X_t) = S_t \end{cases}$$

où σ et μ sont des constantes et les tailles de saut sont normales $\forall i, J_i \sim \mathcal{N}(m, \delta^2)$. Soit donc $Q_t = \sum_{i=1}^{N_t} J_i$ un processus de Poisson composé décrivant le processus de saut. Pour les options vanille, nous disposons de formules de prix explicites, ce qui facilite la calibration. Nous devons calibrer les quatre paramètres du modèle, à savoir la volatilité σ, le paramètre d'occurrence des sauts λ, la moyenne et la variance de la taille des sauts m et δ^2. Le processus s'effectue par le biais d'un problème d'optimisation simple résolu par la programmation quadratique séquentielle des moindres carrés (SLSQP) développée par (Kraft, 1988) et déjà mise en œuvre dans le paquet *Python*'s `scipy`. Les résultats sont présentés dans 3.2.

Résultats et conclusion Le modèle de diffusion à saut de Merton a permis de trouver le prix des options vanille avec une erreur quadratique moyenne relative de 14,89 %. Comparé au modèle Black-Scholes, le modèle de Merton a pu reproduire d'une certaine manière le smile de volatilité comme nous le voyons dans 3.3. La force des processus de diffusion et saut est qu'ils s'adaptent aux asymétries à court terme qui sont généralement plus prononcées puisque l'ajout de sauts capture les variations soudaines dans un futur proche. La volatilité est constante par rapport au temps. Un modèle de Heston sera alors utilisé pour capter la dynamique de la volatilité .

8.3.3 Volatilité stochastique

Le modèle mathématique Soit S_t le processus de prix et V_t le processus de volatilité. La dynamique du modèle de Heston est décrite comme suit.

$$\begin{gathered} dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(S)} \\ dV_t = \kappa(\bar{v} - V_t) dt + \sigma \sqrt{V_t} dW_t^{(V)} \\ d\left\langle W_t^{(S)}, W_t^{(V)} \right\rangle = \rho dt \end{gathered} \tag{8.2}$$

Le terme de drift dans la dynamique de la volatilité est à retour à la moyenne lorsque $\kappa > 0$, avec $\bar{v}$ étant le niveau moyen à long terme de la variance. En fait, si au temps t le processus V_t est supérieur à θ, le terme de dérive poussera la valeur du processus vers le bas.

Calibration du modèle

La calibration du modèle de Heston est plus délicate que celle des autres modèles. Tout d'abord, il y a 5 paramètres à calibrer et la fonction objectif n'est pas connue pour être convexe. Une complexité supplémentaire est ajoutée par la suite en raison de la dépendance de certains paramètres. En fait, σ et κ se compensent, ce qui peut être dû au fait que la fonction objectif est plate pour atteindre l'optimum.
Comme les autres modèles, nous allons essayer de minimiser la différence entre le prix d'une option vanille et celui que l'on trouve sur le marché. Nous utilisons des équations (Cui et al., 2016) car il a simplifié le problème d'optimisation. Pour le résoudre, nous avons utilisé la méthode de Levenberg-Marquardt. La méthode LM est un outil typique pour résoudre un problème de moindres carrés non linéaires, L'étape de recherche est donnée par : $\Delta\boldsymbol{\theta} = \left(\boldsymbol{J}\boldsymbol{J}^{\top} + \mu\boldsymbol{I}\right)^{-1} \boldsymbol{\nabla} f$ En ajustant μ, la méthode passe de la méthode de descente du gradient à la méthode de Gauss-Newton.

Résultats et conclusion En examinant la corrélation entre le prix des actions et la volatilité, nous pouvons voir que, contrairement à de nombreuses études empiriques qui ont documenté une corrélation négative entre le prix des actions et les processus de volatilité (Black, 1976). Dans le cas du bitcoin, le paramètre de corrélation ρ est positif car il y a un nombre plus élevé de call OTM profonds pour la spéculation. Le paramètre positif pourrait également indiquer la présence d'un comportement de noise trading car l'augmentation de la volatilité attire les traders à court terme et donc le prix augmente. Nous avons utilisé le modèle pour déterminer le prix de nos options. Dans l'ensemble, le modèle de Heston a donné de meilleurs résultats que les modèles précédents. L'erreur quadratique moyenne relative du modèle de volatilité stochastique est de 11,84 % pour les 30^{e} de juillet et de 9,76 % pour les 18^{e} d'août. En utilisant les prix de Heston, nous avons dérivé la surface de volatilité implicite dans la fig. 3.6. Dans l'ensemble, le modèle reproduit la surface de volatilité, mais ses performances sont mauvaises pour les échéances courtes. Pour résoudre ce problème, on peut ajouter des sauts à un modèle de volatilité stochastique.

8.3.4 Volatilité stochastique avec co-sauts

Le modèle mathématique

Soit S_t le processus de prix et V_t celui de la volatilité, la dynamique du SVCJ est la suivante :

$$\begin{aligned} d\log S_t &= \mu dt + \sqrt{V_t} dW_t^{(S)} + Z_t^y dN_t \\ dV_t &= \kappa(\theta - V_t)\, dt + \sigma_V \sqrt{V_t} dW_t^{(V)} + Z_t^v dN_t \\ d\left\langle W_t^{(S)}, W_t^{(V)} \right\rangle &= \rho dt \\ \mathrm{P}(dN_t = 1) &= \lambda dt \end{aligned} \tag{8.3}$$

N_t est un processus de saut avec un taux moyen constant d'arrivée par saut moyen constant λ. Les tailles de saut aléatoires sont Z_t^v, Z_t^y. Nous supposons que la taille des sauts aléatoires Z_t^y conditionnelle à Z_t^v est

$$Z_t^y \mid Z_t^v \sim \mathrm{N}\left(\mu_y + \rho_j Z_t^v, \sigma_y^2\right); \quad Z_t^v \sim \exp(\mu_v)$$

Résultats et conclusions

Suivant (Hou et al., 2020), la calibration est effectuée à l'aide des méthodes de Monte Carlo à chaîne de Markov (MCMC). Le processus de calibration a donné les paramètres du tableau 3.1. Le modèle SVCJ s'adapte bien aux données et la significativité des paramètres de saut explique relativement la nécessité de ce modèle. L'évaluation des options vanilles sur crypto-monnaies est effectuée à l'aide de simulations Monte Carlo brutes en utilisant: $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} C(T) \mid \mathcal{F}_t\right]$. Comme prévu, le modèle SVCJ a surperformé les modèles précédents avec un RMSE de 8,12 %. Nous avons pu reproduire la surface de volatilité du SVCJ dans la fig 3.8. Finalement,nous pouvons voir que ce modèle représente bien la surface de volatilité et nous pouvons observer grâce à la surface d'erreur que nous avons relativement résolu le problème d'ajustement du skew à court terme du modèle de Heston. Le prix du Bitcoin est un peu régi par le noise trading. Ce qui nous fait penser qu'il pourrait avoir un comportement fractal. Dans la section suivante, nous étudions cette hypothèse.

8.4 Les crypto-monnaies : Un modèle rugueux s'impose

Les crypto-monnaies et l'hypothèse du marché fractal (FMH) L'hypothèse de l'efficience des marchés implique que le processus de prix reflète toutes les informations disponibles et qu'il est donc impossible pour les acteurs du marché de surperformer systématiquement le marché sur une base ajustée au risque. Dans son livre, Peters a exposé une nouvelle théorie basée sur des études empiriques des différents marchés : L'hypothèse du marché fractal (FMH). Cette théorie stipule que le marché contient de nombreux investisseurs ayant des horizons d'investissement différents et des ensembles d'informations différents. Les investisseurs ayant un horizon à plus long terme fondent leurs décisions sur des informations fondamentales, tandis que les investisseurs à plus court terme se fient aux aspects techniques. Cette structure fournit de la liquidité aux différents agents, ce qui rend le marché stable. En période de turbulence, les horizons d'investissement et les ensembles d'informations s'uniformisent, ce qui assèche la liquidité et fait grimper la volatilité, attirant ainsi davantage d'investisseurs à court terme et de traders de bruit.

Preuve du comportement fractal des crypto-monnaies

Pour capter la structure fractale du marché, nous utilisons l'analyse multi-fractale et l'analyse en ondelettes continues.
L'analyse MF-DFA est utilisée pour mesurer l'exposant de Hurst général $h(q)$. Pour $q = 2$, $h(2)$ correspond à l'exposant de Hurst déjà bien connu H Si $h(2) < 0,5$, la série temporelle est anti-persistante et si $h(2) > 0,5$, elle est persistante. Pour $h(2) = 0.5$, la série temporelle devient une marche aléatoire. En utilisant cette méthode pour le prix et la volatilité journalière du bitcoin que nous avons tracé 4.1. Nous pouvons déja voir un comportement multi-fractale car H dépend de q. Pour les données de prix, H est égal à 0,56. Cela signifie que la courbe du prix du bitcoin montre une persistance en général. En examinant les données 1-min, nous remarquons un coefficient de 0,3 qui montre une certaine réversion de la moyenne et un état turbulent du marché pour les 10 derniers jours de septembre. En examinant le processus des prix pendant cette période, nous pouvons constater un caractère de retour à la moyenne et une légère augmentation de la volatilité historique. En appliquant la même analyse aux données annuelles. Nous remarquons que pour chaque année un comportement multi-fractal est capturé. Nous pouvons également voir que certaines années présentent une anti-persistance. Par exemple, l'année dernière, le processus des rendements quotidiens montre une certaine rugosité, ce qui est cohérent avec les grandes variations de prix que le bitcoin a connu et sa sensibilité aux événements externes et aux nouvelles du marché, ce qui implique la dominance des horizons de négociation à court terme. Un paramètre de Hurst variable, fig 4.3, suggère également que la dynamique du prix des bitcoins varie au fil du temps et qu'un modèle plus général devrait être utilisé pour capturer ce comportement. En appliquant l'analyse multifractale à la volatilité historique, nous voyons qu'elle présente également un comportement multifractal. L'exposant de Hurst est inférieur à $\frac{1}{2}$, ce qui est cohérent avec les résultats observés empiriquement pour d'autres actifs. Ces résultats soutiennent l'utilisation d'un modèle de volatilité rugueuse.

Analyse en ondelettes

La force de l'analyse en ondelettes est sa capacité à visualiser l'information du prix dans les domaines fréquentiel et temporel, fournissant ainsi des informations sur l'évolution des prix à différentes échelles de fréquence et de temps. Selon les arguments du FMH, nous devrions observer une augmentation de la puissance à basse échelle ou à haute fréquence pendant les périodes critiques. En observant la fig. **??**, nous pouvons confirmer ces hypothèses. De plus, nous observons une modification de la structure de la variance à travers les fréquences avant les turbulences en raison de la variation de la structure de l'activité des investisseurs. Une puissance (volatilité) importante et significative est détectée aux hautes fréquences confirmant, ainsi, la dominance du trading à haute fréquence.

8.5 Modèle de volatilité rugueuse

En raison du comportement fractal repéré dans la partie précédente. Nous avons utilisé le modèle rough de Heston décrit comme suit :

$$\begin{gathered} d\log S_t = \mu dt + \sqrt{V_t}dW_t^{(S)} \\ NV_t = \xi_0(t) + \frac{\nu}{\Gamma\left(H+\frac{1}{2}\right)}\int_0^t (t-s)^{H-\frac{1}{2}}\sqrt{V_s}dW_s^{(V)} \\ d\left\langle W_t^{(S)}, W_t^{(V)}\right\rangle = \rho dt \end{gathered} \tag{8.4}$$

où $\nu > 0, H \in \left(0, \frac{1}{2}\right)$. Le caractère rugueux est dû au le noyau $(t-s)^{H-\frac{1}{2}}$. Soit $\alpha = H + \frac{1}{2}$.

Et où $\xi_0(t) = E\left[V_t\right], t \geq 0$, ce qui signifie que la courbe initiale de variance avant est une entrée. Nous avons commencé par simuler le modèle de prix et de volatilité à l'aide de diverses techniques de simulation du mouvement brownien fractal. Cela nous a permis de créer une image d'un marché alimenté à l'aide de ce modèle et de créer ainsi des surfaces de volatilité qui seraient utilisées pour former un réseau neuronal Feed-forward. En utilisant ce réseau neuronal formé, nous avons pu calibrer notre modèle. La calibration a produit différents paramètres qui étaient en ligne avec les autres modèles et qui nous ont aidés à améliorer la précision des prix en diminuant le RMSE à 3,78%. En raison du comportement de multi-fractalité repéré dans la dernière analyse, on pourrait essayer d'utiliser un modèle multi-fractal à l'avenir pour intégrer la dynamique changeante des crypto-monnaies.

8.6 Une réglementation en pleine évolution pour les crypto-actifs

Juin 2021, le comité de Bâle a émis une nouvelle proposition de réglementation pour le marché des crypto-monnaies. Cette proposition était riche pour les actifs tokenisés et les crypto-monnaies avec un système de stabilisation efficace mais était trop restrictive pour les autres crypto-monnaies dont le Bitcoin et l'Etherium et aurait pour conséquence de ne pas impliquer les banques et les institutions financières dans ce marché émergent. Une proposition comprenant plusieurs améliorations a été présentée par les associations financières. Ces améliorations concernent les cryptoactifs du groupe 2, qui comprennent le Bitcoin et les Coins ayant une liquidité suffisante. En utilisant les mesures de la VaR et de la CVaR, nous avons pu confirmer que la proposition de l'ISDA et d'autres institutions concernant les traitements prudentiels des crypto-actifs avait effectivement du sens.

8.7 Conclusion et travaux futurs

Le marché des crypto-monnaies est une mine d'or pour les personnes qui y investissent. Les grandes institutions comme les banques, les fonds spéculatifs ou les compagnies d'assurance s'intéressent de près à ce marché émergent car elles veulent une part du gâteau des rendements phénoménaux de ces dix dernières années. Cependant, ces rendements ont un prix : la volatilité et l'imprévisibilité du prix qui peuvent rendre ces institutions insolvables en un clin d'œil si elles ne peuvent pas gérer les risques liés aux crypto-monnaies. Dans le cadre de ce projet, nous essayons de comprendre ce marché naissant afin de couvrir ses multiples risques. Nous avons montré que des modèles complexes qui incluent la volatilité stochastique et les sauts sont nécessaires pour ajuster la surface de volatilité. Nous avons également montré que le marché présente un comportement fractal et que, par conséquent, un modèle stochastique rough doit être utilisé pour mieux modéliser le prix et la variance sous-jacents. En étudiant les réglementations actuelles, nous pouvons voir qu'elles sont trop restrictives et qu'elles ne peuvent pas réellement aider les banques à s'impliquer dans les crypto-monnaies afin de mettre de l'ordre dans ce marché. En utilisant notre dernier modèle, nous avons pu confirmer la nécessité d'une nouvelle réglementation telle que proposée par les associations financières en septembre 2021. Nous pouvons également confirmer que le comité de Bâle fournit de nouvelles réglementations moins restrictives tout en essayant de surveiller le risque des cryptocurrences. De grandes améliorations sont en cours pour intégrer ce marché émergent mais le travail n'est pas encore terminé.